

CONFORMAL INVARIANTS ASSOCIATED WITH QUADRATIC DIFFERENTIALS

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ABSTRACT. In [14] Z. Nehari developed a general technique for obtaining inequalities for conformal maps and domain functions from contour integrals and the Dirichlet principle. Given a harmonic function with singularity on a domain R , it associates a monotonic functional of subdomains $D \subseteq R$. In the case that R is conformally equivalent to a disk, we extend Nehari's method by associating a functional to any quadratic differential on R with specified singularities. Nehari's method corresponds to the special case that the quadratic differential is of the form $(\partial q)^2$ for a singular harmonic function q on R . Besides being more general, our formulation is conformally invariant, and has a particularly elegant equality statement. As an application we give a one-parameter family of monotonic, conformally invariant functionals which correspond to growth theorems for bounded univalent functions. These generalize and interpolate the Pick growth theorems, which appear in a conformally invariant form equivalent to a two-point distortion theorem of W. Ma and D. Minda.

1. INTRODUCTION

1.1. Statement of results. Let D_1 be conformally equivalent to the unit disk $\mathbb{D} = \{z : |z| < 1\}$ and $D_2 \subseteq D_1$ be a sufficiently regular simply connected domain (we will make this precise below). Let $Q(z)dz^2$ be a quadratic differential such that the boundary of D_1 is network of trajectories of $Q(z)dz^2$. Assume that $Q(z)dz^2$ has finitely many poles in D_1 . In this paper, we define a functional $m(D_1, D_2, Q(z)dz^2)$ with the following properties:

- (1) The functional $m(D_1, D_2, Q(z)dz^2)$ is conformally invariant (Theorem 3.19).
- (2) The functional is monotonic, in the sense that

$$D_3 \subseteq D_2 \subseteq D_1 \Rightarrow m(D_1, D_3, Q(z)dz^2) \leq m(D_1, D_2, Q(z)dz^2)$$

(Corollary 3.24).

- (3) The functional is bounded by zero: $m(D_1, D_2, Q(z)dz^2) \leq 0$ and equality holds for $D_2 = D_1$ (Theorem 3.22). Furthermore, equality holds if and only if $D_1 \setminus D_2$ consists of trajectories of the quadratic differential $Q(z)dz^2$ (Theorem 3.25).

This can be used to give an infinite series of families of inequalities for bounded univalent functions, one for each quadratic differential. Furthermore, these inequalities are given in terms of monotonic coefficient functionals.

The invariants are constructed using a generalization of a technique of Nehari. In this technique, the functionals were generated not by a quadratic differential but rather by a harmonic function with singularities. Our approach results in several improvements which we list here, to be explained in the next section.

- (1) The role of quadratic differentials is made clear (they generate the functional, rather than appear only in a necessary condition for extremality).
- (2) The invariants are manifestly conformally invariant (Theorem 3.19).

- (3) The set of functionals is significantly larger (one for each quadratic differential).
- (4) We explicitly identify the remainder term (that is, the difference between the values of the functional on nested domains) itself as a conformal invariant associated to a quadratic differential (Theorem 3.23).

The reader will likely observe that the results of this paper have straightforward generalizations to hyperbolic Riemann surfaces. In fact, the conformal invariants can be shown to be a special case of a modular invariant on Teichmüller space, as will be shown in a future publication. Here we have restricted to two simply connected domains in order to make a clear and comprehensive connection with bounded univalent functions. We will treat generalizations in future publications.

For every quadratic differential admissible for the disk, the general theorem produces a distinct, sharp estimate on a monotonic functional on the set of bounded univalent functions. Furthermore this functional is automatically conformally invariant. Not unexpectedly, the derivation of an expression for the functional in terms of a conformal map requires a lengthy computation for each choice of quadratic differential. We will give a collection of examples with K. Mather [13]. To illustrate the method here, we give a new one parameter family of growth theorems, generalizing the Pick growth theorem for bounded univalent function. These inequalities in fact interpolate the upper and lower bounds by allowing the zero of the quadratic differential to vary over the boundary. It also leads naturally to a monotonic form of a two-point distortion theorem of Ma and Minda by placing the single and double pole in arbitrary location. The upper and lower bound are obtained by choosing the zero at one of two possible endpoints of the hyperbolic geodesic through the poles.

1.2. Literature and context. In [14] Nehari systematically associated positive monotonic quantities to harmonic functions with singularities, as a method of obtaining inequalities in function theory. This can be thought of as a variant on the method of contour integration, with unifying emphasis on the Dirichlet principle. One advantage of the Nehari's method is that it produces inequalities for higher-order derivatives of mapping functions or domain functions (such as Green's function or Neumann's function) fairly naturally, by choosing the order of the singularity. The second author used this in [21] [22] in order to obtain estimates on higher-order conformal invariants.

However, Nehari's method skips every other order of differentiation. Furthermore it is not manifestly conformally invariant; the conformal invariance must be imposed in ad hoc ways [21]. This paper remedies both problems, by identifying the role of quadratic differentials in Nehari's method, while at the same time extending the method. Nehari's method corresponds to the special case that the quadratic differential is a perfect square of the form $(\partial q)^2$ for a singular harmonic function q . As mentioned above, we restrict to the simply-connected case in this paper.

Conformal invariants are associated with several methods for producing inequalities for mapping functions or domain functions. Three examples are capacitance (e.g. A. Baernstein and A. Solynin [3], P. Duren and J. Pfaltzgraff [7]; or the monograph of V. Dubinin [5]), extremal length (see the monographs of J. Jenkins [9], G. Kuz'mina [11] and A. Vasil'ev [25]) or Dirichlet energy (see Nehari [14] or Dubinin [5]), Of course, given the connections between these conformal invariants [1], there is not always a clear boundary between these methods.

Given the Ahlfors-Beurling theorem connecting Dirichlet energy and extremal length [1, Theorem 4-5], one might expect that the conformal invariance of Nehari's functionals is automatic. This is not the case, since the harmonic function has a singularity and the theorem does not apply. Quadratic differentials of order two are associated with reduced modules; however, as far as the authors are aware there are no examples of conformally invariant reduced modules of higher order. In the literature one often finds the restriction to quadratic differentials with poles of order two or lower, even when conformal invariance is not explicitly demanded. In [13] we will give explicit computations of the conformal invariants given here for various orders of the poles. Reduced modules of higher order are implicit in the so-called general coefficient theorem and the length-area method in general (see e.g. Jenkins [8], [9] and Schmidt [19]); however they are not manifestly conformally invariant.

2. QUADRATIC DIFFERENTIALS AND SIMPLE DOMAINS

2.1. Quadratic differentials: definition and terminology. In this section we review some basic facts about quadratic differentials. In order to properly formulate conformal invariance and admissibility of quadratic differentials, it is necessary to use the formalism of the double and boundary of bordered surfaces. This is also required since we define the invariants by lifting to branched double covers of the disk, which cannot in general be regarded as subsets of \mathbb{C} .

Every Riemann surface in this paper R will be a bordered Riemann surface in the sense of Ahlfors and Sario [2]. For our purposes we may state this as follows. R has a double S ; for each point $p \in \partial R \subset S$ there is an open set U of S and local biholomorphic parameter $\phi : U \rightarrow \mathbb{C}$ such that $\phi(U)$ is an open subset of \mathbb{C} such that (1) $\phi(U \cap \partial R)$ is an open interval of \mathbb{R} , (2) if $q \in U$ then the conjugate point q^* is in U and (3) $\phi(q^*) = \overline{\phi(q)}$. Thus ϕ is obtained from $\phi|_{U \cap R}$ by Schwarz reflection. We call such a chart a boundary coordinate and the image points $z \in \phi(U)$ (as a function of points on the surface R) as a boundary parameter. For an arbitrary chart ϕ on R we refer to image points as local parameters. Observe that the boundary of R is an analytic curve in the double.

A quadratic differential on a bordered Riemann surface R is a meromorphic 2-differential on R ; locally it can be written $Q(z)dz^2$ for a local parameter z [10].

Definition 2.1. Let α be a quadratic differential on a bordered Riemann surface R and let $\Gamma : (a, b) \rightarrow R$ be a smooth curve. We say that Γ is a trajectory if given any point p in the image of Γ and local parameter z , the local representations $z = \gamma(t)$ of Γ and $Q(z)dz^2$ of α satisfy

$$(2.1) \quad Q(\gamma(t)) \cdot \gamma'(t)^2 < 0.$$

If $\Gamma : [a, b] \rightarrow \mathbb{C}$ is a continuous curve whose restriction to (a, b) is a trajectory, then we will also refer to Γ as a trajectory. Similarly for half-open intervals.

If $\Gamma : (a, b) \rightarrow \partial R$ is a smooth curve, then we say that Γ is a trajectory if α extends complex analytically to an open neighbourhood of the image of Γ in the double of R and Γ is a trajectory in the sense above on the double. Similarly for closed or half-open intervals.

It is easily checked that this definition is independent of the choice of local parameter. We will also refer to the image of Γ as a trajectory.

We say that a quadratic differential is admissible for R if, for all but finitely many points p on the border, there is a boundary parameter in an open neighbourhood of every boundary

point such that the local expression $Q(z)dz^2$ satisfies (2.1) for some parametrization $\gamma(t)$ of a portion of the real axis containing $z(p)$. We will also say that R is admissible for the quadratic differential in this case.

Definition 2.2. Let R be a bordered Riemann surface and α be a quadratic differential on R . We say that α is admissible for R if all but finitely many points p on the boundary ∂R are in the image of a trajectory $\Gamma : (a, b) \rightarrow \partial R$ of α .

Definition 2.3. Let α be a quadratic differential on a Riemann surface S and $f : R \rightarrow S$ be a local biholomorphism. The “pull-back” of α to R under f is defined by, for a local representation $Q(w)dw^2$ of α and $w = f(z)$ of f ,

$$f^*(Q(w)dw^2) = Q(f(z))f'(z)^2dz^2.$$

Remark 2.4. If R and S are bordered Riemann surfaces, and α extends meromorphically to a neighbourhood of \overline{S} in the double of S , and f is holomorphic on ∂R , then the pull-back can be extended to the boundary.

Remark 2.5. It is immediately evident that if γ is a trajectory of $f^*(\alpha)$ if and only if $f \circ \gamma$ is a trajectory of α , since

$$Q(f(\gamma(t)))f'(\gamma(t))^2 \left(\frac{d\gamma(t)}{dt} \right)^2 = Q(f \circ \gamma(t)) \left(\frac{df \circ \gamma}{dt} \right)^2.$$

This fact extends to trajectories on the boundary when f and Q are sufficiently regular.

Using the Schwarz reflection principle it is easy to see that if $Q(w)dw^2$ is admissible for R_2 then the pull-back under a conformal bijection is admissible for R_1 . Note that this is not necessarily true if f is not a bijection. A conformal bijection of R_1 to R_2 extends to a conformal bijection of the doubles S_i of R_i , $i = 1, 2$, and an admissible quadratic differential on R_i extends uniquely to a quadratic differential on the doubles by reflection. Thus, the statement that an admissible quadratic differential has zeros and poles on the boundary has a conformally invariant meaning.

In this paper we will be concerned entirely with simply connected Riemann surfaces conformally equivalent to the disk and their double covers with finitely many branch points. If R is a simply connected bordered surface, then R and its double are conformally equivalent to \mathbb{D} and $\overline{\mathbb{C}}$ respectively, and the boundary can be identified with $\partial\mathbb{D}$. Of course R is conformally equivalent to any simply connected domain Ω in the plane which is not \mathbb{C} . In either case we can represent the quadratic differential globally on Ω as $Q(z)dz^2$ for the global parameter z on \mathbb{C} . If we choose $\Omega = \mathbb{D}$ then $Q(z)$ extends to a rational function on $\overline{\mathbb{C}}$.

Remark 2.6. According to the above definitions, it makes sense to say that a quadratic differential is admissible for Ω even when $\partial\Omega$ is highly irregular.

Remark 2.7. If Ω is represented as a planar domain bounded by a piecewise analytic Jordan curve, then a quadratic differential $Q(z)dz^2$ is admissible for Ω if and only if the boundary segments $\gamma(t)$ satisfy $Q(\gamma(t))\gamma'(t)^2 < 0$ for the local meromorphic extension of $Q(z)$ across the boundary curve.

However, if the boundary is piecewise analytic but not a Jordan curve (e.g. so that for some $p \in \partial\Omega$ there is an open neighbourhood U of p such that $U \cap \Omega$ has two disjoint

components) then $Q(z)dz^2$ might not have a consistent extension from the two “sides” of $U \cap \partial\Omega$. However, the two sides of $U \cap \partial\Omega$ correspond to distinct analytic arcs in the double. Thus this problem is avoided in our formulation above. This subtlety is one of the chief reasons we require the formalism of borders and doubles in this paper.

Finally we will need the following elementary theorem [16, Theorem 8.1].

Theorem 2.8. *Let $Q(z)dz^2$ be a quadratic differential on an open connected set $U \subseteq \mathbb{C}$.*

- (1) *If $Q(z_0) \neq 0$, then there exists a neighbourhood V of z_0 in U and a biholomorphism $\phi : V \rightarrow W \subset \mathbb{C}$ such that for $w = \phi(z)$ we have $Q(z)dz^2 = dw^2$ (that is, $\phi'(z)^2 = Q(z)$).*
- (2) *If Q has a zero of order $n > 0$ at z_0 , then there exists a neighbourhood V of z_0 in U and a biholomorphism $\phi : V \rightarrow W \subset \mathbb{C}$ such that for $w = \phi(z)$ we have $Q(z)dz^2 = w^n dw^2$ (that is, $\phi'(z)^2 \phi(z)^n = Q(z)$).*

It is also possible to classify the poles, but we will not have need of this. We will refer to points where $Q(z_0)$ has neither a zero nor a pole as regular points.

2.2. Simple domains. In the next section we will define invariants depending on pairs of nested simply-connected domains and a quadratic differential. In this subsection, we specify the regularity of the inner domain. In fact, the functionals can be extended to much more irregular domains; however, to avoid lengthening this paper needlessly we will not pursue this here. By conformal invariance, there is no restriction on the regularity of the outer domain (see Remark 2.6).

From now on, a “conformal disk” is a bordered Riemann surface conformally equivalent to the disk. The reader can replace “conformal disk” with $\mathbb{D} = \{z : |z| < 1\}$ everywhere, if she is willing to take on faith that the results are conformally invariant.

Definition 2.9. Let R be a conformal disk, with double \hat{R} . An open connected set $D \subseteq R$ is called “simple” if (1) it is simply connected (2) there exists a quadratic differential α on the closure of D in \hat{R} which is admissible for D , and (3) this quadratic differential has no poles on ∂D .

This class of domains has the following properties: (1) the functionals are easily defined on it without introducing analytic difficulties, (2) it includes the extremal domains, and (3) it is dense in the set of all simply-connected proper subsets of \mathbb{C} in a certain sense.

The conformal disk R is itself simple, as can be seen by identifying R and its double with \mathbb{D} and $\overline{\mathbb{C}}$ and considering the quadratic differential dz^2/z^2 on $\overline{\mathbb{C}}$. For a conformal disk R , it is clear that a simple domain $D \subseteq R$ is itself a conformal disk. The condition that D be simple imposes a further condition on the regularity of ∂D as it appears in R . In principle, one should always say D is simple *with respect to R* , although for brevity we will usually drop this last phrase.

The boundary of a simple domain D consists of a finite collection of analytic arcs, joined at a finite number of “vertices”. Let \mathfrak{V} denote the set of zeros of a quadratic differential α on ∂D together with those points which are an endpoint of only a single trajectory arc. The latter type of point may be zeros or regular points of α . We call this the set of vertices of D .

Theorem 2.10. *Let D be a simple domain in a conformal disk R . For a quadratic differential α admissible for D , the set of vertices $\mathfrak{V} = \{v_1, \dots, v_n\}$ on ∂D is finite. The complement*

$\partial D \setminus \mathfrak{V}$ consists of finitely many analytic arcs. Furthermore, at each vertex, there is a neighbourhood which intersects only finitely many of these arcs, which meet at the vertex at equally spaced angles.

Proof. The fact that there are finitely many vertices follows from the fact that the quadratic differential is meromorphic on the compact closure of R , the compactness of ∂D , and the fact that the quadratic differential is not identically zero. The second claim follows from applying part (1) of Theorem 2.8 in local coordinates, after observing that at any regular point the trajectory of α through a regular point in a sufficiently small neighbourhood is the image of an interval of the imaginary axis under ϕ^{-1} . The final claim follows from part (2) of Theorem 2.8. \square

Remark 2.11. It is possible that some of the boundary curves of D are arcs of ∂R . These are analytic curves in the double of R . If R is identified with \mathbb{D} , then these are arcs of $\partial \mathbb{D}$.

It is elementary that simple domains are dense in the set of simply connected proper subsets of \mathbb{C} in the following sense.

Proposition 2.12. *Let R be a bordered Riemann surface conformally equivalent to the disk, and let D be any simply connected subset of R . Let $f : \mathbb{D} \rightarrow D$ be a conformal bijection. There is a sequence of holomorphic maps $f_n : \mathbb{D} \rightarrow D$ such that $f_n(\mathbb{D}) \subset f_{n+1}(\mathbb{D})$ is a conformal bijection for all n , $f_n \rightarrow f$ uniformly on compact sets, and each D_n is a simple domain bounded by a single analytic Jordan curve.*

Proof. Let $f_n(z) = f((1 - 1/n)z)$, let $Q(z)dz^2 = dz^2/z^2$ and set

$$Q_n(z)dz^2 = \frac{(f_n^{-1})'(z)^2 dz^2}{f_n^{-1}(z)^2}$$

on D_n . Clearly $f_n(\partial \mathbb{D})$ is an analytic Jordan curve, and since f_n has a bijective holomorphic extension to a neighbourhood of $\overline{\mathbb{D}}$, by Remark 2.5 $Q_n(z)dz^2$ meet the conditions of Definition 2.2. \square

3. CONFORMAL INVARIANTS ASSOCIATED TO QUADRATIC DIFFERENTIALS

3.1. Harmonic pairs on double covers adapted to quadratic differentials. In order to define the conformal invariants, we will need a covering on which the quadratic differential has a single-valued square root. We first define such covering and then show that it has the desired properties.

Definition 3.1. Let D be a simple domain in a conformal disk R , and let α be a quadratic differential admissible for D (at least one exists by definition). We say that $\pi : \tilde{D} \rightarrow D$ is a cover adapted to α if it is a double-sheeted cover of D with a branch point of order two at each odd-order zero and pole of α .

Recall that there are at most finitely many zeros and poles. If there are no poles or zeros of odd order, then \tilde{D} consists of two disjoint sheets biholomorphic to D . In that case, any curve in D has a two distinct lifts, each lying entirely in one sheet. If there is at least one odd order pole or zero, then a closed curve γ in $D \setminus \{z_1, \dots, z_k, p_1, \dots, p_m\}$ lifts to a closed curve in \tilde{D} if and only if the sum of the winding numbers of γ with respect to the points z_i, p_j is even.

The double cover adapted to a quadratic differential is uniquely determined up to a conformal map.

Proposition 3.2. *Let D be a simple domain in a conformal disk R . Let \tilde{D}_1 and \tilde{D}_2 denote two covers of D adapted to α . There exists a conformal map $\phi : \tilde{D}_1 \rightarrow \tilde{D}_2$.*

Proof. If there are no poles or zeros of odd order, then \tilde{D}_1 and \tilde{D}_2 both consist of two disjoint sheets each of which is biholomorphic to D , and the claim follows immediately.

Now assume that there is at least one zero or pole of odd order. Fix $z_0 \in D$ and let p_i and q_i denote the two preimages of z_0 under π_i , $i = 1, 2$. The map $\pi_1 : \tilde{D}_1 \rightarrow D$ has a single-valued lift to a map $\hat{\pi}_1 : \tilde{D}_1 \rightarrow \tilde{D}_2$ such that $\hat{\pi}_1(p_1) = p_2$ (by lift, we mean that, $\pi_2 \circ \hat{\pi}_1 = \pi_1$). To see this it is enough to show that π induces a map from the fundamental group of the covering \tilde{D}_1 into that of \tilde{D}_2 . To this end observe that every non-trivial element of the fundamental group at z_0 of the covering \tilde{D}_1 can be represented as a lift of a closed curve γ such that the sum of the winding numbers around the odd zeros and poles of Q is even. However, this is precisely the condition that γ be a representative of the covering group of \tilde{D}_2 . This proves the claim. Similarly, π_2 has a lift to a map $\hat{\pi}_2 : \tilde{D}_2 \rightarrow \tilde{D}_1$ such that $\hat{\pi}_2(p_2) = p_1$. Both maps $\hat{\pi}_i$ are holomorphic.

Since $\pi_2 \circ \hat{\pi}_1 = \pi_1$, we have that $\pi_2 \circ (\hat{\pi}_1 \circ \hat{\pi}_2) = \pi_1 \circ \hat{\pi}_2 = \pi_2$. Thus $\hat{\pi}_1 \circ \hat{\pi}_2$ is a lift of the identity; since $\hat{\pi}_1 \circ \hat{\pi}_2(p_2) = p_2$ by uniqueness of lifts it must be the identity. Similarly $\hat{\pi}_2 \circ \hat{\pi}_1$ is the identity. Setting $\phi = \hat{\pi}_1$ we have proven the proposition. \square

The condition that a quadratic differential be admissible for D implies that the primitive of its square root on the double cover has a single-valued real part, at least in a doubly-connected domain near the boundary ∂D . The next proposition formulates this precisely.

Given a quadratic differential α admissible for D and a double cover $\pi : \tilde{D} \rightarrow D$ adapted to α , since π is a local biholomorphism away from branch points, $\pi^*\alpha$ is a well-defined quadratic differential on R minus the branch points. It is easy to see that $\pi^*\alpha$ remains bounded at branches and therefore extends to a quadratic differential on \tilde{D} .

Theorem 3.3. *Let D be a conformal disk and let α be an admissible quadratic differential for D . Let \tilde{D} be a double-cover of D adapted to α . Let $F : \mathbb{D} \rightarrow D$ be a conformal bijection. Let \mathfrak{B} be the set of branch points in \tilde{D} of the cover $\pi : \tilde{D} \rightarrow D$.*

- (1) *There is a well-defined meromorphic one-form β on \tilde{D} such that $\beta^2 = \pi^*(\alpha)$.*
- (2) *The one-form α has a multi-valued holomorphic primitive x on $\tilde{D} \setminus \mathfrak{B}$. For some $0 < r < 1$, x has a single-valued real part on $\pi^{-1}F(r < |z| < 1)$. The real part of the primitive x extends to a well-defined harmonic function on any domain of the form $D \setminus \Omega$ where Ω is a simply connected domain, containing the odd order poles of α whose closure is in D .*
- (3) *$q = \text{Re}(x)$ extends continuously to a constant function on $\partial \tilde{D}$.*

Remark 3.4. If D is a conformal disk (in fact any bordered Riemann surface), it is easily seen that if \tilde{D} is a double cover of D all of whose branch points are in the interior, then \tilde{D} is also a bordered Riemann surface. For any border chart ϕ in a neighbourhood of $p \in \tilde{D}$, $\phi \circ \pi$ is a border chart near each of the two points in $\pi^{-1}(p)$ for locally biholomorphic choices of π^{-1} . We will use this in the following proof.

Proof. Using pull-backs, we may assume that D is the unit disk \mathbb{D} and therefore $\alpha = Q(z)dz^2$ in terms of the global parameter z . (In fact, by admissibility $Q(z)$ must be a rational function on $\overline{\mathbb{C}}$).

To prove (1), first we observe that at any $p \in \tilde{D} \setminus \mathfrak{B}$, there are two square roots of $\pi^*Q(z)dz^2$ defined in a neighbourhood of p as follows. Assume first that $\pi(p)$ is not a pole or zero of $Q(z)dz^2$. If ζ is a local holomorphic coordinate in a neighbourhood of p and $\pi^*Q(z)dz^2 = h(\zeta)d\zeta^2$ we can locally set $\beta = \sqrt{h(\zeta)}d\zeta$, and there are precisely two choices of $\sqrt{h(\zeta)}$. Using the transformation property of quadratic differentials, it is easily verified that this pair of locally defined one forms is independent of the choice of local coordinates. If now $\pi(p)$ is a zero or pole of $Q(z)dz^2$, then it must have even order, since $p \notin \mathfrak{B}$. Since π is a covering (and therefore has non-zero derivative in local coordinates), p is also a zero or pole of $\pi^*Q(z)dz^2$ of the same order. In local coordinates ζ we have that $\pi^*Q(z)dz^2 = (\zeta - \zeta(p))^{2n}H(\zeta)d\zeta^2$ for some non-vanishing holomorphic function H and we again have two square roots $(\zeta - \zeta(p))^n\sqrt{H(\zeta)}$ in a neighbourhood of p . Note that on a sufficiently small open set U containing p , the locally defined one-form β is $\pi^*(\delta)$ for some one-form δ on $\pi(U)$.

Now assume that $\gamma : [0, 1] \rightarrow D$ is a closed curve winding once around a odd order zero or pole, say p , but winding around no other odd order zero or pole. The lift of γ is such that $\gamma(0)$ and $\gamma(1)$ lie on different sheets. If we continue a choice of square root of $\pi^*(Q(z)dz^2)$ along γ , the corresponding one-form δ defined on a neighbourhood of $\gamma([0, 1])$ on D picks up a sign change each time it winds around p . Thus in general the sign change of the continuation of δ along any closed curve γ is $(-1)^k$ where k is the sum of the winding numbers of γ around the odd order zeros and poles. The sign change of the corresponding continuation $\beta = \pi^*\delta$ is the same.

Now fixing a point $p_0 \in \tilde{D} \setminus \mathfrak{B}$ we make a choice of square root in a neighbourhood, and analytically continue it to $\tilde{D} \setminus \mathfrak{B}$. It needs to be shown that this continuation is single valued. Let Γ be a closed curve in \tilde{D} through p_0 . Since the sum of the winding numbers of $\pi \circ \Gamma$ around odd order zeros and poles is even, the continuation of β along Γ is single-valued by the previous paragraph. Thus β is well-defined on $\tilde{D} \setminus \mathfrak{B}$.

To see that β extends meromorphically to a branch point $b \in \tilde{D}$, observe that there is a local coordinate z say in a neighbourhood of b , and a local coordinate w in a neighbourhood of $\pi(b)$, in which π has the form $w = z^2$ (with $w = 0$ and $z = 0$ corresponding to $\pi(b)$ and b respectively). If α in local coordinates has the form $Q(w)dw^2$ then $\pi^*\alpha$ has the form $4Q(z^2)z^2dz^2$, so $\pi^*\alpha = z^{2m}h(z)dz^2$ for some integer m and holomorphic h such that $h(0) \neq 0$. Thus $\pi^*\alpha$ has a well-defined square root $z^m\sqrt{h(z)}dz$ in a neighbourhood of b , and this must agree with β on the punctured neighbourhood. This proves (1).

We prove (2) and (3) simultaneously. Let r be large enough that $\pi^{-1}(F(r < |z| < 1))$ contains no zeros and poles of $Q(z)dz^2$. On this domain define the (generally multi-valued) analytic function

$$x(\zeta) = \int_{\zeta_0}^{\zeta} \beta.$$

By Remark 3.4, $\pi^*(Q(z)dz^2)$ continues analytically to $\partial\tilde{D}$, and hence so do β and x .

We now show that $\operatorname{Re} x$ is constant on $\partial\tilde{D}$. Let $\gamma(t)$ parameterize a portion of the boundary curve of $\partial\tilde{D}$. We have that

$$Q(\pi \circ \gamma(t)) \left(\frac{d(\pi \circ \gamma)}{dt}(t) \right)^2 \leq 0$$

and thus for either choice of square root of Q in a neighbourhood of $\gamma(t)$ we have

$$\operatorname{Re} \left(\sqrt{Q(\pi \circ \gamma(t))} \cdot \frac{d(\pi \circ \gamma)}{dt}(t) \right) = 0$$

and therefore $\pi^*(Q(z)dz^2)$ has the same property with respect to γ . Thus the one-form β evaluated in a direction tangent to $\partial\tilde{D}$ is pure imaginary. In particular, $\operatorname{Re}(x)$ is constant on $\partial\tilde{D}$.

It is clear that x extends analytically to any domain of the specified type, although of course it might be multi-valued. It remains to prove that the real part of x is single valued. Now β (which equals $x'(\zeta)d\zeta$ in coordinates ζ) is a single-valued holomorphic one-form on the doubly-connected domain $\pi^{-1}(F(r < |z| < 1))$, and by the previous paragraph the period of β on this domain is pure imaginary. This proves that $\operatorname{Re}(x)$ is single valued on this domain. \square

Remark 3.5. It is immediately seen that β extends to a meromorphic differential on the double of \tilde{D} .

Let ∂ denote the differential operator given in local coordinates by

$$\partial h = \frac{\partial h}{\partial z} dz.$$

In the future we will sometimes denote the one-form β by ∂x or locally by $x'(\zeta)d\zeta$. Note that because $\operatorname{Re}(x)$ may have singularities, the maximum principle cannot be applied so it need not be constant on D . Clearly x is determined uniquely up to an additive constant.

In the literature, $x \circ \pi^{-1}$ is called the canonical or straightening map of $Q(z)dz^2$. The terms can refer to either a single-valued choice of $x \circ \pi^{-1}$ on the domain D minus branch cuts or the multi-valued function. In the proof of Theorem 3.3 it appeared that if a curve is a trajectory of a quadratic differential then the corresponding function $\operatorname{Re}(x)$ is constant on its lift. The converse is also true. In a local parameter z , observe that x and $Q(z)dz^2$ are related by

$$(3.1) \quad Q(z)dz^2 = \left(\frac{\partial}{\partial z} x \circ \pi^{-1}(z) \right)^2 dz^2.$$

If α denotes the quadratic differential, then we can globally write $\alpha = (\partial(x \circ \pi^{-1}))^2$.

Proposition 3.6. *Let D be a conformal disk and α be a quadratic differential admissible for D . Let \tilde{D} be a double cover adapted to α with branch points $\mathfrak{B} \subseteq \tilde{D}$ and let x be the multi-valued meromorphic function on $\tilde{D} \setminus \mathfrak{B}$ of Theorem 3.3; that is $\alpha = (\partial(x \circ \pi^{-1}))^2$. A curve $\gamma(t)$ is a trajectory of α if and only if $\operatorname{Re}(x)$ is constant on $\pi^{-1} \circ \gamma$ for any local choice of π^{-1} .*

Proof. By conformal invariance we can assume that D is the disk \mathbb{D} , and thus we have a global parameter z with $\alpha = Q(z)dz^2$. The first claim follows directly from the proof of Proposition

3.6. On the other hand, if $\operatorname{Re}(x)$ is constant on $\gamma(t)$ then for some local determination of π^{-1} , setting $\Gamma = \pi \circ \gamma$ we have

$$\begin{aligned} 0 &= \frac{d}{dt} \operatorname{Re} [x(\gamma(t))] = \frac{d}{dt} \operatorname{Re} [x \circ \pi^{-1} \circ \Gamma(t)] \\ &= \operatorname{Re} \left[\frac{\partial x \circ \pi^{-1}}{\partial z} \circ \Gamma \cdot \frac{d\Gamma}{dt} \right] \end{aligned}$$

Thus

$$Q(\Gamma(t)) \frac{d\Gamma^2}{dt} = \left[\frac{\partial x \circ \pi^{-1}}{\partial z} \circ \Gamma \cdot \frac{d\Gamma}{dt} \right]^2 \leq 0.$$

□

Let D_1 and D_2 be simple domains such that $D_2 \subseteq D_1$, and let $Q(z)dz^2$ be an admissible quadratic differential for D_1 . Assume that all poles of $Q(z)dz^2$ which are contained in the interior of D_1 are also contained in the interior of D_2 . Let $\pi : \tilde{D}_1 \rightarrow D_1$ be a double cover of D_1 adapted to $Q(z)dz^2$. Let x_1 be the function guaranteed by Theorem 3.3 and let $q_1(z) = \operatorname{Re}(x_1(z))$. Assume that the additive constant of x_1 is chosen so that $q_1 = 0$ on $\partial\tilde{D}_1$.

Definition 3.7. Let D_1 be a conformal disk and let D_2 be a simple subdomain of D_1 . Let α be a quadratic differential admissible for D_1 all of whose poles are in D_2 . Let \tilde{D}_1 be a double cover adapted to α and $\tilde{D}_2 = \pi^{-1}(D_2)$. Let x_1 be one of the primitives of the square root of α on \tilde{D}_1 and set $q_1 = \operatorname{Re}(x_1)$, with the additive constant chosen so that $q_1 = 0$ on $\partial\tilde{D}_1$. Let u be the unique harmonic function on \tilde{D}_2 such that $u = q_1$ on $\partial\tilde{D}_2$ and set $q_2 = q_1 - u$.

- (1) We call $\pm(q_1, q_2)$ the harmonic pair induced by $(D_1, D_2, Q(z)dz^2)$.
- (2) We call $\alpha_2 = (\partial(x_2 \circ \pi^{-1}))^2$ the quadratic differential on D_2 induced by α . In a local coordinate z , we have that α_2 can be written $Q_2(z)dz^2 = 4 \left(\frac{\partial q_2 \circ \pi^{-1}(z)}{\partial z} \right)^2 dz^2$.

Remark 3.8 (convention for disconnected cover). In the case that $Q(z)dz^2$ has no double poles or zeros, the double cover \tilde{D}_1 has two connected components, as observed above. In this case, we adopt the following convention. The primitive x_1 is chosen so that for any fixed point $z \in D_1$, the two values of $q_1 = \operatorname{Re} x_1$ at $\pi^{-1}(z)$ differ by a sign. With this restriction, there are two (rather than four) possible choices of harmonic pair $\pm(q_1, q_2)$ on \tilde{D}_1 , in agreement with the case that \tilde{D}_1 is connected.

Remark 3.9. Applying Theorem 3.3, we see that q_2 is single-valued near $\partial\tilde{D}_2$ in the sense of the Proposition. The function u is single-valued and non-singular on \tilde{D}_2 .

Of course it must be verified that the induced harmonic pair and quadratic differential are well-defined. Observe that q_2 is uniquely determined by x_1 and D_2 . Furthermore, x_1 (and hence q_1) is determined up to a sign; that is given one such function q_1 , $\pm q_1$ are the only two functions satisfying the definition, and furthermore for the unique non-trivial deck transformation $g : \tilde{D}_1 \rightarrow \tilde{D}_1$ we have that $q_1(g(z)) = -q_1(z)$. Clearly if q_2 is the harmonic function on \tilde{D}_2 associated with q_1 as in Definition 3.7, then $-q_2$ is the harmonic function on \tilde{D}_2 associated with $-q_1$. Thus (q_1, q_2) and $(-q_1, -q_2)$ are the only pairs satisfying Definition 3.7. Thus the harmonic pair $\pm(q_1, q_2)$ is well-defined.

Furthermore we have the following.

Proposition 3.10. *Let D_1 be a conformal disk and D_2 be simple domain in D_1 . Let α_1 be an admissible quadratic differential for D_1 . Assume that all poles of α_1 are contained in the interior of D_2 .*

- (1) *The quadratic differential α_2 on D_2 induced by α_1 is well-defined. Furthermore, α_2 is admissible for D_2 .*
- (2) *If $\pm(q_1, q_2)$ is the harmonic pair associated with (D_1, D_2, α_1) then q_2 is the singular harmonic function on \tilde{D}_2 associated to α_2 as in Theorem 3.3.*

Proof. Let g be the deck transformation of \tilde{D}_1 . We claim that $q_2(g(z)) = -q_2(z)$. To see this, let u be the solution to the Dirichlet problem on $\partial\tilde{D}_2$ with boundary values equal to q_1 on $\partial\tilde{D}_2$, and recall that $q_2 = q_1 - u$. We have that for all $z \in \partial\tilde{D}_2$, $u(g(z)) = q_1(g(z)) = -q_1(z) = -u(z)$. Thus $u(g(z)) + u(z)$ is zero on $\partial\tilde{D}_2$ and so $u(g(z)) = -u(z)$ for all $z \in \tilde{D}_2$. In particular $q_2(g(z)) = -q_2(z)$ for all $z \in \tilde{D}_2$.

Now fix an open set $V \subseteq D_2$, with a local coordinate z , where V is chosen so that $\pi^{-1}(V)$ has precisely two disjoint components U and \hat{U} and π has biholomorphic inverses π^{-1} and $\hat{\pi}^{-1}$ on V . We thus have that $\pi^{-1} = g \circ \hat{\pi}^{-1}$ on V . By the previous paragraph,

$$\frac{\partial q_2 \circ \pi^{-1}}{\partial z}(z) dz = -\frac{\partial q_2 \circ \hat{\pi}^{-1}}{\partial z}(z) dz$$

for all $z \in U$. Thus in local coordinates the expression

$$Q_2(z) dz^2 = 4 \left(\frac{\partial q_2 \circ \pi^{-1}}{\partial z}(z) \right)^2 dz^2$$

for α_2 is independent of the local choice of π^{-1} . This shows that the quadratic differential on D_2 induced by α is well-defined.

Thus to show that α_2 is admissible for D_2 we need only show that the boundary is a trajectory. Let x_2 be the multi-valued meromorphic function on $\tilde{D}_2 \setminus \mathfrak{B}$ whose real part is q_2 . By the Cauchy-Riemann equations, in local coordinates we have

$$Q_2(z) dz^2 = 4 \left(\frac{\partial q_2 \circ \pi^{-1}}{\partial z} \right)^2 dz^2 = \left(\frac{\partial x_2 \circ \pi^{-1}}{\partial z} \right)^2 dz^2;$$

That is, $\alpha_2 = (\partial(x_2 \circ \pi^{-1}))^2$. Thus α_2 is admissible by Proposition 3.6. This also proves the second claim. \square

Proposition 3.11. *Let D_1 be a conformal disk and D_2 be a simple domain in D_1 . Let α_1 be a quadratic differential admissible for D_1 .*

- (1) *If \tilde{D}_1 and \hat{D}_1 are two distinct covers of D_1 adapted to α_1 , and $g : \tilde{D}_1 \rightarrow \hat{D}_1$ is a conformal map, then the corresponding multi-valued holomorphic functions x_1 and \hat{x} satisfy $\hat{x}_1 \circ g = x_1$. Similarly $\hat{q}_i \circ g = q_i$ for $i = 1, 2$.*
- (2) *The quadratic differential on D_2 induced by α_1 is independent of the choice of cover adapted to α_1 . In particular, the induced quadratic differential is well-defined.*

Proof. The first claim follows from Proposition 3.2. The second is immediate. \square

It is also elementary that

Proposition 3.12. *If D_1 is a conformal disk, α_1 is an admissible quadratic differential for D_1 , and \tilde{D}_1 is a cover adapted to α_1 , then the harmonic pair induced by (D_1, D_1, α_1) is $\pm(q_1, q_1)$. Furthermore the induced differential $Q_2(z)dz^2$ on D_2 equals $Q(z)dz^2$.*

Finally, the harmonic pairs and induced quadratic differential also have a kind of transitivity property.

Proposition 3.13. *Let D_1 be a conformal disk and let D_2 and D_3 be simple domains in D_1 satisfying $D_3 \subseteq D_2 \subseteq D_1$. Let α_1 be an admissible quadratic differential for D_1 , all of whose poles are in D_3 . Let $\pi_1 : \tilde{D}_1 \rightarrow D_1$ be a double cover adapted to $Q_1(z)dz^2$. For $i = 1, 2$ let α_i be the quadratic differentials on D_i induced by α_1 .*

- (1) $\pi_1|_{\tilde{D}_2}$ is a double cover of D_2 adapted to α_2 .
- (2) If $\pm(q_1, q_i)$ are the harmonic pairs induced by (D_1, D_i, α_1) for $i = 1, 2$ then $\pm(q_2, q_3)$ is the harmonic pair induced by (D_2, D_3, α_2) .
- (3) α_3 is the quadratic differential on D_3 induced by α_2 .

Proof. We can assume that $D_1 = \mathbb{D}$, so that we have a global parameter z . Let $\alpha_i = Q_i(z)dz^2$ for $i = 1, 2, 3$. We first prove (1). It is immediate that x_2 has the same poles as x_1 , and of the same order, since $q_1 - q_2$ is non-singular. Thus $Q_1(z)dz^2$ and $Q_2(z)dz^2$ have odd order poles at precisely the same points.

Since Q_2 is a perfect square at all points which are not branch points, Q_2 cannot have an odd order zero unless Q_1 . So we need only show that if $Q_1(z)dz^2$ has an odd order zero at a point $w_0 \in D_2$, then $Q_2(z)dz^2$ also has an odd order zero at w_0 . Let g denote the unique deck transformation of order two on \tilde{D}_1 . Assume that $z_0 \in D_2 \subseteq D_1$ is an odd order zero of $Q_1(z)dz^2$. Thus $w_0 = \pi_1^{-1}(z_0)$ is a branch point of π_1 and hence a fixed point of g . Since $q_2(g(w)) = -q_2(w)$ for all w we also have that in a local coordinate w

$$(3.2) \quad \frac{\partial q_2}{\partial w}(w_0)dw = \frac{\partial q_2}{\partial w}(g(w_0))dw = -\frac{\partial q_2}{\partial w}(w_0)dw$$

so w_0 is a zero of $\partial q_2 / \partial z$ and hence $z_0 = \pi_1(w_0)$ is a zero of

$$(3.3) \quad Q_2(z)dz^2 = 4 \left(\frac{\partial q_2 \circ \pi_1^{-1}}{\partial z}(z) \right)^2 dz^2.$$

This proves (1).

Next we prove (2). By Proposition 3.10 q_2 is the singular harmonic function associated to $Q_2(z)dz^2$ as in Theorem 3.3. Since $q_3 = 0$ on $\partial \tilde{D}_3$ and $q_2 - q_3 = q_2 - q_1 + q_1 - q_3$ is harmonic on \tilde{D}_3 , it follows that $\pm(q_2, q_3)$ is the harmonic pair induced by $(D_2, D_3, Q_2(z)dz^2)$.

(3) is an immediate consequence of (2). □

3.2. Definition of the conformal invariants. Given a smooth function h on a Riemann surface R , we define a differential operator $*dh$ as follows. On any domain $U \subset R$ with local parameter $z = x + iy$ and real-valued function h define the one-form

$$*dh = \frac{\partial h}{\partial x}dy - \frac{\partial h}{\partial y}dx.$$

$*$ is often called the Hodge star operator.

The expression

$$(3.4) \quad *dh = \operatorname{Re} \left(\frac{2}{i} \frac{\partial h}{\partial z} dz \right)$$

is often convenient. For example, it is easily computed that if $z = g(w)$ is a conformal map defined on an open neighbourhood of a contour γ then for any real maps h_1 and h_2

$$(3.5) \quad \begin{aligned} \int_{\gamma} h_1 \circ g * d(h_2 \circ g) &= \operatorname{Re} \left(\int_{\gamma} h_1 \circ g \frac{2}{i} \frac{\partial h_2}{\partial z} \circ g \cdot g'(w) dw \right) \\ &= \operatorname{Re} \left(\int_{g \circ \gamma} h_2 \frac{2}{i} \frac{\partial h_1}{\partial z} dz \right) \\ &= \int_{g \circ \gamma} h_1 * dh_2. \end{aligned}$$

This amounts to the same thing as the fact that the Hodge star operator is independent of the choice of local parameter z .

Finally observe that in a local parameter $z = x + iy$,

$$d * dh = \Delta h \cdot dx \wedge dy = -2i \partial \bar{\partial} h \cdot d\bar{z} \wedge dz.$$

where ∂ and $\bar{\partial}$ are the standard d- and d-bar operators, and

$$\Delta h = \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}$$

is the Laplacian. It is easily verified that $d * dh$ is independent of the choice of local parameter.

Remark 3.14. Note that if γ is a positively oriented curve in a local coordinate U , n denotes the normal directed to the right of travel and ds denotes infinitesimal Euclidean arc length, then

$$(3.6) \quad \int_{\gamma} *dh = \int_{\gamma} \frac{\partial h}{\partial n} ds.$$

This is the notation employed by Nehari. We will use this notation ahead when computing examples in the plane.

We may now define the conformal invariants.

Definition 3.15 (Module associated to (D, D_1, α)). Let D_1 be a conformal disk and $D_2 \subseteq D_1$ be a simple domain in D_1 . Let α be a quadratic differential which is admissible for D_1 . Let $\pi : \tilde{D}_1 \rightarrow D_1$ be a double cover of D_1 adapted to $Q(z)dz^2$. Let $\pm(q_1, q_2)$ be the harmonic pair induced by (D_1, D_2, α) on \tilde{D}_1 . We call

$$(3.7) \quad m(D_1, D_2, \alpha) = \int_{\partial \tilde{D}_2} q_1 * dq_2$$

the module of (D_1, D_2, α) .

The meaning of this integral requires clarification. The issues are as follows. If D_2 is bordered by a single analytic curve, then this is an ordinary contour integral. Consider however the following example. Let D_2 be the disk with a single radial slit, so that \tilde{D}_2 also possesses at least one slit Γ say. It is clear that q_2 will extend continuously to Γ (in fact all of $\partial \tilde{D}_2$ in general). However $*dq_2$ will have two distinct extensions, one for each “side”

of the slit. It is natural in this case to interpret the integral along the boundary of $\partial\tilde{D}_2$ as containing one integral for each extension of $*dq_2$, with opposite orientations. We will make this precise below.

It also needs to be established that this module is well-defined. There are two issues: first, that the integral converges, and second, that the module depends only on D_1 , D_2 and α as the notation suggests, and not on the choice of double cover. Although the convergence is elementary, there are many details to address, so we have relegated the proof to an appendix in order not to interrupt the flow of the paper. The remainder of this section will be devoted to establishing that the integral is well defined, and also that the module is conformally invariant.

We now clarify the meaning of the integral (3.7). The function $*dq_2$ has a one-sided extension in the following sense. For any point $p \in \partial\tilde{D}_2$ such that $\pi(p)$ is not a zero of α , by 2.8 there is an open disk B centred on p such that $B \setminus \partial\tilde{D}_2$ consists of two connected components U and V , each of which is either in \tilde{D}_2 or disjoint from it. At least one of U or V must be contained in \tilde{D}_2 ; assume that it is U . Since $*dq_2$ is a harmonic form and $\partial\tilde{D}_2 \cap B$ is an analytic curve (it is in particular locally of the form $\text{Re}(-2i\partial h/\partial z dz)$ for some holomorphic h by (3.4), it has a harmonic extension to some open neighbourhood of $U \cup (\partial\tilde{D}_2 \cap B)$. If V is also in \tilde{D}_2 , the same argument applies to V ; however, the two extensions do not in general agree on their overlap. We abbreviate the above paragraph by saying that $*dq_2$ has a harmonic extension to $\partial\tilde{D}_2$ from one or two sides in a neighbourhood of any point p such that $\pi(p)$ is not a zero.

Let $\mathfrak{V} \subset \partial D_2$ consist of all points in ∂D_2 which are either zeros of α or terminal points of trajectories of α (which may in fact be regular points). We call this the set of vertices of ∂D_2 . We will also call the set $\pi^{-1}(\mathfrak{V})$ the vertices of $\partial\tilde{D}_2$.

Let Γ be an arc of $\partial\tilde{D}_2$ whose endpoints are vertices, and containing no other vertices. We adopt the convention that the endpoints are not in Γ (or any analytic arcs below). By the argument above, one of two possibilities hold: either (1) for all $p \in \Gamma$, there is an open set B containing p such that $B \cap \tilde{D}_2$ has precisely two connected components and precisely one of these components is in \tilde{D}_2 , or (2) for all $p \in \Gamma$, there is an open set B containing p such that $B \cap \tilde{D}_2$ has precisely two connected components and both are in \tilde{D}_2 . If Γ satisfies the first condition we call it a one-sided boundary arc and if it satisfies the second condition we call it a two-sided boundary arc. The same reasoning and terminology holds for sub-arcs of ∂D_2 with this property.

Definition 3.16. Let D_2 be a simple subset of a conformal disk D_1 . A complete set of maximal boundary arcs of \tilde{D}_2 is a collection γ_i , $i = 1, \dots, 2m$ oriented analytic arcs with the following properties:

- (1) $\cup_{i=1, \dots, m} \gamma_i \cup \pi^{-1}(\mathfrak{V}) = \partial\tilde{D}_2$
- (2) the image of γ_i joins two vertices, and contains no other vertices
- (3) for every one-sided arc Γ of $\partial\tilde{D}_2$, there is precisely one γ_i with the same image; γ_i is positively oriented with respect to \tilde{D}_2
- (4) for every two-sided arc Γ of $\partial\tilde{D}_2$, there are precisely two curves γ_i and γ_j with the same image as Γ ; γ_i and γ_j have opposite orientations.

It is clear that the collection of γ_i are determined uniquely up to ordering. Also, given a point on a two-sided arc γ_i and a neighbourhood B of p such that $B \cap \tilde{D}_2$ has two connected components, γ_i will be positively oriented with respect to precisely one of the components of $B \cap \tilde{D}_2$. We extend $*dq_2$ from this component to γ_i . With this convention we define the integral (3.7) as follows.

Definition 3.17. Let D_2 be a simple domain in a conformal disk D_1 . Provided that each integral converges, we define

$$\int_{\partial \tilde{D}_2} q_1 * dq_2 = \sum_{i=1}^m \int_{\gamma_i} q_1 * dq_2$$

where $\gamma_1, \dots, \gamma_m$ are a complete collection of maximal boundary arcs satisfying properties (1) - (4) above, and on each arc γ_i we choose the harmonic extension of $*dq_2$ determined by the orientation of γ_i .

The convergence of this integral is guaranteed by the following theorem. The proof is given in the Appendix 5.

Theorem 3.18. Let D_2 be a simple domain in a conformal disk D_1 . Let α be a quadratic differential which is admissible for D_1 and let $\pm(q_1, q_2)$ be the induced harmonic pair, and let $\gamma_1, \dots, \gamma_m$ be a complete set of maximal boundary arcs of $\partial \tilde{D}_2$. Each integral

$$(3.8) \quad \int_{\gamma_i} q_1 * dq_2$$

converges. Furthermore, letting $F : \mathbb{D} \rightarrow D_2$ be a conformal bijection and setting C_r to be the curve $|z| = r > 1$ with positive orientation, we have

$$\int_{\partial \tilde{D}_2} q_1 * dq_2 = \lim_{r \nearrow 1} \int_{\pi^{-1}(F(C_r))} q_1 * dq_2.$$

Note that $\pi^{-1}(F(C_r))$ is one or two closed analytic curves. This theorem also verifies that Definition 3.17 is sensible.

Now we prove that the module is well-defined. Recall that if α is a quadratic differential admissible for a conformal disk D_1 say, and $g : E_1 \rightarrow D_1$ is a conformal bijection, then the pull back $g^*\alpha$ preserves trajectories. Thus α is admissible for D_1 if and only if $g^*\alpha$ is admissible for E_1 .

Now assume that $D_2 \subseteq D_1$ and $E_2 \subseteq E_1$ are simple with respect to D_1 and E_1 respectively. Assume also that $g(E_2) = D_2$. Let β be the one-form on \tilde{D}_1 such that $\beta^2 = \pi^*(g^*\alpha)$ and δ be the one-form on \tilde{E}_1 such that $\delta^2 = \pi^*\alpha$, whose existence is guaranteed by Theorem 3.3, and let x and y be their primitives respectively. Let $\tilde{g} : \tilde{D}_1 \rightarrow \tilde{E}_1$ be the lift of g to the double cover. It is immediately seen that if $\beta = b(z)dz$ and $\delta = c(z)dz$ in a local coordinate, then $b(z) = c(\tilde{g}(z))\tilde{g}'(z)$ (possibly after switching the sign of δ). Thus $x = y \circ \tilde{g}$.

Set $q_1 = \text{Re}(x)$ and $p_1 = \text{Re}(y)$, so that (q_1, q_2) and (p_1, p_2) are each a harmonic pair (with a definite choice of sign) induced by $(D_1, D_2, g^*\alpha)$ and (E_1, E_2, α) respectively. It is clear that if $p_1 - p_2$ is harmonic then $p_1 \circ \tilde{g} - p_2 \circ \tilde{g}$ is harmonic. Thus since $q_1 = p_1 \circ \tilde{g}$ we have $(q_1, q_2) = (p_1, p_2)$.

Next, observe that by Proposition 3.2, given two distinct covers \tilde{D}_1 and \hat{D}_1 of D_1 , there is a conformal map $\phi : \tilde{D}_1 \rightarrow \hat{D}_1$. The integral 3.7 is invariant under ϕ by (3.5), so the module

$m(D_1, D_2, Q(z)dz^2)$ is well-defined. Similarly, applying (3.5) we see that the integral (3.7) is invariant under composition by \tilde{g} . Thus we have proven the following theorem.

Theorem 3.19. *Let D_1 be a conformal disk, and D_2 be simple subdomain. Let $g : D_1 \rightarrow E_1$ be a conformal bijection and $E_2 = g(D_2)$. If α is admissible for E_1 then*

$$m(D_1, D_2, g^* \alpha) = m(E_1, E_2, \alpha).$$

In other words, the modules are conformally invariant.

Remark 3.20 (convention for disconnected cover, part two). In the case that α has no zeros and poles of odd order, \tilde{D}_1 is disconnected. In that case there is a meromorphic one-form β on D_1 such that $\beta^2 = \alpha$, and the contour integral defining the module reduces to a contour integral over ∂D_2 as follows. Let y_1 denote the primitive on D_1 , and $p_1 = \text{Re}(y_1)$; similarly define p_2 such that $p_1 - p_2$ is harmonic and $p_2 = 0$ on ∂D_2 . It is clear that (p_1, p_2) is the restriction to D_1 of one of the harmonic pairs $\pm(q_1, q_2)$. Thus we have

$$m(D_1, D_2, \alpha) = 2 \int_{\partial D_2} p_1 * dp_2.$$

We will see ahead that this special case includes the functionals defined by Nehari.

3.3. Monotonicity theorems. In this section we prove the main result of the paper.

We recall Green's identities, which we will need in their general form on Riemann surfaces (in order to apply them on double covers). Let u and v be functions on a bordered Riemann surface R bounded by analytic curves, which are C^2 functions on the closure of R . We then have

$$(3.9) \quad \int_{\partial R} v * du = \iint_R v \wedge *u + \iint_R v \cdot d * du.$$

which implies another Green's identity

$$(3.10) \quad \int_{\partial R} (v * du - u * dv) = \iint_R (v \cdot d * du - u \cdot d * dv).$$

In local coordinates $z = x + iy$ observe that

$$du \wedge *dv = \frac{1}{2i} (u_x v_x + u_y v_y) d\bar{z} \wedge dz$$

so in particular if $u = v$ the integral is the familiar Dirichlet energy of u and must be non-negative.

If there is a global coordinate z on R , denoting infinitesimal arc length by ds and the unit outward normal by n , these have the form

$$\int_{\partial R} v \frac{\partial u}{\partial n} ds = \iint_R \nabla u \cdot \nabla v dA + \iint_R v \Delta u dA$$

and

$$\int_D \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) ds = \iint_D (v \Delta u - u \Delta v) dA$$

where dA is the measure $d\bar{z} \wedge dz/2i$.

Lemma 3.21. *Let D_1 be a conformal disk and $D_2 \subseteq D_1$ be a simple subdomain. Let α be a quadratic differential admissible for D_1 , and let $\pm(q_1, q_2)$ be the corresponding harmonic pair on D_1 . Let D_3 be any open subset of D_2 which contains all of the poles of α . For any double cover \tilde{D}_1 adapted to α ,*

$$\iint_{\tilde{D}_i \setminus \tilde{D}_3} dq_i \wedge *dq_i < \infty$$

for $i = 1, 2$.

Proof. Fix i . Let Ω be a domain bounded by analytic Jordan curves, whose closure is contained in \tilde{D}_3 and which contains all the singularities of q_i . Let $F : \mathbb{D} \rightarrow D_i$ be a conformal bijection and C_r the curve $|z| = r$ with positive orientation. By Green's identity (3.9) we have

$$\begin{aligned} \iint_{\tilde{D}_i \setminus \tilde{D}_3} dq_i * dq_i &\leq \iint_{\tilde{D}_i \setminus \Omega} dq_i * dq_i = \lim_{r \nearrow 1} \iint_{\pi^{-1} \circ F(\{|z| < r\}) \setminus \Omega} dq_i * dq_i \\ &= \lim_{r \nearrow 1} \int_{\pi^{-1} \circ F(C_r)} q_i * dq_i - \int_{\partial \Omega} q_i * dq_i \\ &= - \int_{\partial \Omega} q_i * dq_i \end{aligned}$$

where the last equality follows from Theorem 3.18. Since q_i is harmonic on an open set containing $\partial \Omega$ the final integral exists. \square

Theorem 3.22 (Positivity). *Let D_1 be a conformal disk and $D_2 \subseteq D_1$ a simple subdomain, and let α be admissible for D_1 . Then*

$$m(D_1, D_2, \alpha) \leq 0.$$

Proof. Assume for now that D_2 is bounded by an analytic curve in D_1 . Let (q_1, q_2) be one of the harmonic pairs induced by (D_1, D_2, α) . Finally let

$$w(p) = \begin{cases} q_1(p) & p \in \tilde{D}_1 \setminus \tilde{D}_2 \\ q_1(p) - q_2(p) & p \in \tilde{D}_2. \end{cases}$$

Observe that this is well-defined since $q_1 - q_2$ is single-valued and q_1 is single-valued on $\tilde{D}_1 \setminus \tilde{D}_2$ by Theorem 3.3. In that case, using Green's identity (3.9),

$$\begin{aligned} \iint_{\tilde{D}_1} dw \wedge *dw &= \iint_{\tilde{D}_1 \setminus \tilde{D}_2} dw \wedge *dw + \iint_{\tilde{D}_2} dw \wedge *dw \\ &= \int_{\partial \tilde{D}_1} q_1 * dq_1 - \int_{\partial \tilde{D}_2} q_1 * dq_1 + \int_{\partial \tilde{D}_2} (q_1 - q_2) * d(q_1 - q_2) \\ (3.11) \quad &= - \int_{\partial \tilde{D}_2} q_1 * dq_2. \end{aligned}$$

Since the left hand side is greater than or equal to zero this completes the proof in the case that the boundary of D_2 is analytic.

For the general case, let $F_i : \mathbb{D} \rightarrow D_i$ be conformal bijections for $i = 1, 2$. In the computation above replace $\partial \tilde{D}_i$ with $\pi^{-1} F_i(C_r)$, replace $\tilde{D}_1 \setminus \tilde{D}_2$ with the region bounded by $\pi^{-1} \circ F_1(C_r)$ and $\pi^{-1} \circ F_2(C_r)$, and \tilde{D}_2 by $\pi^{-1} \circ F_2(\{|z| < r\})$. Letting $r \nearrow 1$ and applying Theorem 3.18 and Lemma 3.21 completes the proof. \square

We also have the following theorem, which says that the modules have a kind of transitivity property.

Theorem 3.23. *Let D be a conformal disk and let D_1 and D_2 be simple domains such that $D_2 \subseteq D_1 \subseteq D$. Let α be a quadratic differential which is admissible for D and assume that all of the poles of α are contained in D_2 . Let $\pm(q, q_i)$ be the harmonic pairs induced by (D, D_i, α) for $i = 1, 2$. Then*

$$m(D, D_2, \alpha) - m(D, D_1, \alpha) = \int_{\partial \tilde{D}_2} q_1 * dq_2.$$

Thus if α_1 is the quadratic differential on D_1 induced by α then

$$m(D, D_2, \alpha) - m(D, D_1, \alpha) = m(D_1, D_2, \alpha_1).$$

Proof. Assuming that D_2 is bounded by an analytic curve, we compute

$$\begin{aligned} I &= -m(D, D_1, \alpha) - \int_{\partial \tilde{D}_2} q_1 * dq_2 + m(D, D_2, \alpha) \\ &= - \int_{\partial \tilde{D}_1} q * dq_1 - \int_{\partial \tilde{D}_2} q_1 * dq_2 + \int_{\partial \tilde{D}_2} q * dq_2 \\ &= \int_{\partial \tilde{D}_1} (q_1 - q) * dq_1 - \int_{\partial \tilde{D}_2} (q_1 - q) * dq_2 \end{aligned}$$

where we have used the fact that $q_1 = 0$ on $\partial \tilde{D}_1$. By Green's identity (3.10)

$$\begin{aligned} \int_{\partial \tilde{D}_1} (q_1 - q) * dq_1 &= \int_{\partial \tilde{D}_1} (q_1 - q) * dq_1 - \int_{\partial \tilde{D}_1} q_1 * d(q_1 - q) \\ &= \int_{\partial \tilde{D}_2} (q_1 - q) * dq_1 - \int_{\partial \tilde{D}_2} q_1 * d(q_1 - q) \end{aligned}$$

since $q_1 - q$ and q_1 are harmonic on $\tilde{D}_1 \setminus \tilde{D}_2$. So

$$\begin{aligned} I &= \int_{\partial \tilde{D}_2} (q_1 - q) * dq_1 - \int_{\partial \tilde{D}_2} q_1 * d(q_1 - q) - \int_{\partial \tilde{D}_2} (q_1 - q) * dq_2 \\ &= \int_{\partial \tilde{D}_2} (q_1 - q) * d(q_1 - q_2) - \int_{\partial \tilde{D}_2} (q_1 - q_2) * d(q_1 - q) = 0 \end{aligned}$$

by Green's identity. The general case is handled by approximating with analytic curves as in the proof of Theorem 3.22.

The final claim follows from Proposition 3.13. □

Theorems 3.22 and 3.23 immediately imply that the higher-order reduced modules are monotonic in the following sense.

Corollary 3.24 (Monotonicity). *If D , D_1 and D_2 are simple domains satisfying $D_2 \subseteq D_1 \subseteq D$ and α is a quadratic differential admissible for D then*

$$m(D, D_2, \alpha) \leq m(D, D_1, \alpha).$$

3.4. The case of equality. Equality in Theorem 3.22 occurs only if α is also admissible for D_2 .

Theorem 3.25. *Equality occurs in Theorem 3.22 if and only if $D_1 \setminus D_2$ consists of a network of trajectories of α .*

Proof. By equation (3.11) equality holds if and only if

$$\iint_{\tilde{D}_1} |\nabla w|^2 dA = 0.$$

So in particular we have that

$$\iint_{\tilde{D}_1 \setminus \tilde{D}_2} |\nabla q_1|^2 dA = 0;$$

since q_1 is not constant this can only occur if $\tilde{D}_1 \setminus \tilde{D}_2$ has measure zero. We already know that \tilde{D}_2 is bounded by a network of analytic arcs, since D_2 is simple. Thus $\tilde{D}_1 \setminus \tilde{D}_2$ is a network of a finite number of analytic arcs $\gamma_1, \dots, \gamma_k$. We must also have that

$$\iint_{\tilde{D}_2} |\nabla(q_1 - q_2)|^2 dA = 0$$

which can only occur if $q_1 - q_2$ is constant (and hence zero, since $\partial\tilde{D}_1$ and $\partial\tilde{D}_2$ must have points in common). Since $q_2 = 0$ on γ_i for $i = 1, \dots, k$ we must also have that $q_1 = 0$ on those curves. By Proposition 3.6 the curves $\pi \circ \gamma_i$ are trajectories of α . \square

Remark 3.26. Assuming only that D_1 is a conformal disk one can extend the functional $m(D_1, D_2, \alpha)$ to general simply connected domains $D_2 \subseteq D_1$ in various ways (for example, by writing it in terms of derivatives of Green's function [21] or approximating the simply connected domain by simple ones). This is easily done for arbitrary quadratic differentials with finitely many poles and zeros, although we will not show this here.

Furthermore, by conformal invariance we can without loss of generality assume that $D_1 = \mathbb{D}$ and $D_2 = f(\mathbb{D})$ for a conformal map f . Applying the Schiffer variational technique [18, Theorem II.29] for bounded univalent functions shows that the extremal function for an extended functional maps onto the disk minus trajectories of a quadratic differential; that is, the inner domain is simple. Thus it can be shown that Theorem 3.25 holds for the extended functional.

We will not pursue the general extension and equality case here. Instead, in Section 4, we will give specific functionals which extend by inspection to functionals for arbitrary simply connected D_2 , which are continuous with respect to uniform convergence on compact sets. Furthermore they can easily be shown to satisfy the conditions of [18, Theorem II.29], so Theorem 3.25 holds for these specific functionals.

We also have the following elementary consequence of monotonicity and boundedness.

Corollary 3.27. *Let D_1 be a conformal disk and $D_2 \subseteq D_1$ a simple domain. Let α be a quadratic differential which is admissible for D_1 . If $m(D_1, D_2, \alpha) = 0$, then $m(D_1, D_3, \alpha) = 0$ for all domains D_3 such that $D_2 \subseteq D_3 \subseteq D_1$.*

Proof. This follows immediately from Corollary 3.24 and Corollary 3.22, once we know that D_3 is simple. However that follows immediately from the fact that D_2 is simple as a consequence of Theorem 3.25. \square

3.5. Relation to Nehari's general inequality. Nehari's monotonicity theorem [14] says the following, in the simply connected case. (The domains in [14] are assumed to have analytic boundary curves, whereas here they might be only piecewise analytic).

Theorem 3.28 (Nehari monotonicity theorem, simply connected case). *Let D be a conformal disk and D_1 and D_2 be simple domains such that $D_2 \subseteq D_1 \subseteq D$. Let S be a harmonic function on D , except possibly for finitely many singularities in D_2 . Let p_i be the unique functions on D_i such that $p_i = 0$ on ∂D_i and $S + p_i$ is harmonic on D_i . Then*

$$\int_{\partial D_2} S \frac{\partial p_2}{\partial n} ds \geq \int_{\partial D_1} S \frac{\partial p_1}{\partial n} ds.$$

Thus defining the functional

$$M(D, D_1, S) = \int_{\partial D_1} S \frac{\partial p_1}{\partial n} ds$$

we have that $M(D, D_2, S) \geq M(D, D_1, S)$ whenever $D_2 \subseteq D_1 \subseteq D$. We call this functional the ‘‘Nehari functional’’. If we let p be the unique function on D such that $S + p$ is harmonic and $p = 0$ on ∂D , then the lower bound of this functional is

$$M(D, D, S) = \int_{\partial D} S \frac{\partial p}{\partial n} ds.$$

It can be shown using Green's identity [24] that

$$M(D, D_1, S) - M(D, D, S) = \int_{\partial D_1} S \frac{\partial p_1}{\partial n} ds - \int_{\partial D} S \frac{\partial p}{\partial n} ds = - \int_{\partial D_1} p \frac{\partial p_1}{\partial n} ds.$$

Since $p - p_1 = (p + S) - (S + p_1)$ is harmonic on D_1 we can rewrite this as

$$(3.12) \quad M(D, D_1, S) - M(D, D, S) = M(D, D_1, -p).$$

Note that $M(D, D, -p) = 0$.

This leads to the following Proposition.

Proposition 3.29. *Let D be a conformal disk and $D_1 \subseteq D$ a simple subdomain, and S be a singularity function on D as in Nehari's theorem 3.28. Let $M(D, D_1, S)$ be the resulting Nehari functional. Letting p be the unique function on D such that $p = 0$ on ∂D and $S + p$ is harmonic, setting $\hat{S} = -p$ we have $M(D, D_1, S) = M(D, D_1, \hat{S}) + C$ where C is independent of D_1 . With this choice of singularity function $M(D, D, \hat{S}) = 0$. Furthermore*

$$\alpha = 4 (\partial p)^2$$

is admissible for D and

$$M(D, D_1, \hat{S}) = -\frac{1}{2} m(D, D_1, \alpha).$$

Proof. To be consistent with Nehari's notation, without loss of generality we choose $D = \mathbb{D}$ and set

$$\alpha = Q(z) dz^2 = 4 \frac{\partial p}{\partial z} dz^2.$$

The first two claims were proven above (equation (3.12) and immediately following).

To see that $Q(z)dz^2$ is admissible for D , observe that since p is zero on ∂D

$$\frac{\partial p}{\partial s} ds = 0$$

along ∂D , and thus $\partial p/\partial z$ is pure imaginary on ∂D . Thus $Q(z)dz^2 \leq 0$ on ∂D , that is ∂D is a trajectory of $Q(z)dz^2$.

The final claim follows from Remark 3.20 once we observe that $p - p_1 = p + S - (p_1 + S)$ is harmonic on D_1 and $p_1 = 0$ on ∂D_1 ; hence $\pm(p, p_1)$ is the harmonic pair associated with $(D, D_1, Q(z)dz^2)$. \square

In other words, without loss of generality, we can assume that the singularity function in Nehari's theorem is the primitive of the square root of a quadratic differential. This shows that Corollary 3.24 significantly generalizes the simply connected case of Nehari's theorem 3.28 by removing the requirement that the quadratic differential be a perfect square. It is evident that the techniques of this paper can also be used to extend Nehari's theorem in the case of finitely connected domains.

Remark 3.30. In Schiffer's method, the quadratic differential is generated by taking a functional derivative of a fixed functional. The differential is not completely determined by the functional and depends on the extremal function; furthermore admissibility of the map is necessary but not sufficient for extremality. In this paper, the quadratic differential is fixed, and admissibility is necessary and sufficient.

Remark 3.31. It is natural to ask whether the functionally derivative of $m(D, D_1, \alpha)$ can be written in terms of α . This is indeed true in the case that α is a perfect square with pole at the origin [24].

4. GROWTH THEOREMS

In this section we use Theorem 3.22 and Corollary 3.24 to derive a family of growth theorems for bounded univalent functions.

4.1. Preliminary computations. In this section we collect some computations that will be useful in the proof of the main application.

First, we will write the functional in a form which is easier to compute. In the following, we express the contour integrals in terms of a local parameter z . For ease of presentation, we assume that there is a global parameter z ; if the parameter is only local, the expressions are still valid along some subcontour. For any real function h we introduce the notation

$$\frac{\partial h}{\partial s} ds = \frac{\partial h}{\partial x} dx + \frac{\partial h}{\partial y} dy.$$

We may thus write

$$(4.1) \quad 2 \frac{\partial h}{\partial z} dz = \frac{\partial h}{\partial s} ds + i \frac{\partial h}{\partial n} ds$$

where ds is infinitesimal arc length and n is the normal to the right of direction of motion.

We will find a more computable expression for the conformal invariants. Let D be a conformal disk and D_1 be a simple domain such that $D_1 \subseteq D$; let α be a quadratic differential canonically admissible for D . Let \tilde{D} denote the double cover of D adapted to $Q(z)dz^2$. Let

$\pm(q, q_1)$ denote the harmonic pairs induced by (D, D_1, α) , and x, x_1 be the analytic functions whose single-valued positive real parts are q, q_1 respectively.

Since $q_1 = \operatorname{Re}(x_1) = 0$ on $\partial\tilde{D}_1$,

$$\begin{aligned} m(D, D_1, \alpha) &= \int_{\partial\tilde{D}_1} q \frac{\partial q_1}{\partial n} ds = \operatorname{Re} \left(\int_{\partial\tilde{D}_1} x \frac{\partial q_1}{\partial n} ds \right) \\ &= \operatorname{Re} \left(\int_{\partial\tilde{D}_1} (x - x_1) \frac{\partial q_1}{\partial n} ds \right). \end{aligned}$$

Again using the fact that $q_1 = 0$ on \tilde{D}_1 , we have that

$$\frac{\partial q_1}{\partial s} ds = 0$$

along $\partial\tilde{D}_1$ so

$$\frac{\partial q_1}{\partial n} ds = \frac{2}{i} \frac{\partial q_1}{\partial z} dz$$

along ∂D_1 by (4.1). Furthermore by the Cauchy-Riemann equations

$$\frac{\partial x_1}{\partial z} = 2 \frac{\partial q_1}{\partial z}$$

so

$$m = \operatorname{Re} \left(\frac{1}{i} \int_{\partial\tilde{D}_1} (x - x_1) \frac{\partial x_1}{\partial z} dz \right).$$

Furthermore, since $x - x_1$ and its z -derivative are non-singular analytic functions,

$$\int_{\partial\tilde{D}_1} (x - x_1) \left(\frac{\partial x_1}{\partial z} - \frac{\partial x}{\partial z} \right) dz = 0.$$

Thus we have the identity

$$(4.2) \quad m(D, D_1, Q(z)dz^2) = \int_{\partial\tilde{D}_1} q \frac{\partial q_1}{\partial n} ds = \operatorname{Re} \left(\frac{1}{i} \int_{\partial\tilde{D}_1} (x - x_1) \frac{\partial x}{\partial z} dz \right).$$

4.2. A general two-point growth theorem. We consider the following quadratic differential, for $r > 0$:

$$(4.3) \quad Q(z)dz^2 = -e^{-i\psi} \frac{(e^{i\psi} + z)^2}{z^2(z - r)(z - 1/r)} dz^2.$$

For $z = e^{i\theta}$ we have that

$$Q(z) = re^{-2i\theta} \frac{2 \cos(\psi/2 - \theta/2) + 2}{|1 - re^{i\theta}|^2}$$

so $Q(z)dz^2$ is admissible for \mathbb{D} . If we can compute this module, then by conformal invariance we know the module of any quadratic differential with a double pole and simple zero in the interior, and double zero on $\partial\mathbb{D}$.

We require a double cover of \mathbb{D} branched at $r > 0$ on which to compute x ; we choose $\tilde{\mathbb{D}} = \mathbb{D}$ and

$$\begin{aligned} \pi : \tilde{\mathbb{D}} &\rightarrow \mathbb{D} \\ \zeta &\mapsto T(\zeta^2) \end{aligned}$$

where

$$T(w) = \frac{w + r}{1 + rw}.$$

Next we pull back the quadratic differential to the cover:

$$\begin{aligned} \pi^*(Q(z)dz^2) &= -\frac{e^{-i\psi} (e^{i\psi} + (\zeta^2 + r)/(1 + r\zeta^2))^2}{\left(\frac{\zeta^2 + r}{1 + r\zeta^2}\right)^2 \left(\frac{\zeta^2 + r}{1 + r\zeta^2} - r\right) \left(\frac{\zeta^2 + r}{1 + r\zeta^2} - \frac{1}{r}\right)} \cdot \frac{4\zeta^2(1 - r^2)^2}{(1 + r\zeta^2)^4} d\zeta^2 \\ (4.4) \quad &= 4r(1 + re^{i\psi})^2 e^{-i\psi} \frac{(T(e^{i\psi}) + \zeta^2)^2}{(1 + r\zeta^2)^2 (\zeta^2 + r)^2} d\zeta^2. \end{aligned}$$

Theorem 4.1. *Let $Q(z)dz^2$ be given by (4.3). Then for a one-to-one conformal mapping $f : \mathbb{D} \rightarrow \mathbb{D}$ such that $f(0) = 0$ and $r \in f(\mathbb{D})$ we have that*

$$(4.5) \quad m(\mathbb{D}, f(\mathbb{D}), Q(z)dz^2) = 4\pi \operatorname{Re} \left[-e^{i\psi} \log \left(\frac{w_0 f'(0)}{f(w_0)} \frac{1 - |f(w_0)|^2}{1 - |w_0|^2} \right) \right] + \log \left(\frac{1 + |f(w_0)|}{1 - |f(w_0)|} \frac{1 - |w_0|}{1 + |w_0|} \right)$$

where $w_0 = f^{-1}(r)$, the first term uses the unique branch of logarithm such that $\log(w_0 f'(0)/f(w_0)) \rightarrow 0$ as $w_0 \rightarrow 0$ and the second term uses the principal branch.

Proof. By (4.4) we have that

$$\begin{aligned} x'(\zeta) &= 2\sqrt{r}e^{-i\psi/2} \frac{(1 + re^{i\psi})(T(e^{i\psi}) + \zeta^2)}{(1 + r\zeta^2)(\zeta^2 + r)} \\ &= \frac{2\sqrt{r}e^{i\psi/2}}{\zeta^2 + r} + \frac{2\sqrt{r}e^{-i\psi/2}}{1 + r\zeta^2} \\ (4.6) \quad &= \frac{ie^{i\psi/2}}{\zeta + i\sqrt{r}} - \frac{ie^{i\psi/2}}{\zeta - i\sqrt{r}} + \frac{\sqrt{r}e^{-i\psi/2}}{1 + i\sqrt{r}\zeta} + \frac{\sqrt{r}e^{-i\psi/2}}{1 - i\sqrt{r}\zeta}. \end{aligned}$$

Thus setting $\beta = ie^{i\psi/2}$ we have

$$(4.7) \quad x(\zeta) = \beta \log \frac{\zeta + i\sqrt{r}}{\zeta - i\sqrt{r}} - \bar{\beta} \log \frac{1 - i\sqrt{r}\zeta}{1 + i\sqrt{r}\zeta}$$

where we choose the principal branch of the logarithm. This requires some justification. Observing that $\zeta \mapsto (\zeta + i\sqrt{r})/(\zeta - i\sqrt{r})$ maps $\{|\zeta| > \sqrt{r}\}$ onto the right half plane, and $\zeta \mapsto (1 - i\sqrt{r}\zeta)/(1 + i\sqrt{r}\zeta)$ maps $\{|\zeta| < 1/\sqrt{r}\}$ onto the right half plane, so this function is well-defined. It is easily checked that it is a primitive of x' . Since $x(1) = 0$, it follows from the fact that $\partial\mathbb{D}$ is a trajectory of $Q(z)dz^2$ that $\operatorname{Re}(x) = 0$ on $\partial\mathbb{D}$. It is also easily shown directly using $\bar{\zeta} = 1/\zeta$.

Now let D_1 be a simple subdomain of \mathbb{D} containing 0 and r , and \tilde{D}_1 be $\pi^{-1}(D_1)$. We will find an explicit formula for x_1 in terms of conformal maps. We use the following notation. Let $f : \mathbb{D} \rightarrow D_1$ be a conformal bijection such that $f(0) = 0$. Let $G = S^{-1} \circ f^{-1} \circ T$ for

$$S(w) = \frac{w + w_0}{1 + \overline{w_0}w}.$$

It can then be checked that $G(-r) = -w_0$ and $G(0) = 0$. Furthermore G is a conformal bijection from $T^{-1}(D_1)$ onto \mathbb{D} . Finally set $\tilde{G} = \sqrt{G(\zeta^2)}$, so that \tilde{G} is a conformal map from \tilde{D}_1 onto $\tilde{\mathbb{D}}$ such that $\tilde{G}(0) = 0$.

It follows immediately from the analysis of x and the properties of \tilde{G} that

$$x_1(\zeta) = \beta \log \left[\frac{\tilde{G}(\zeta) - \tilde{G}(-i\sqrt{r})}{\tilde{G}(\zeta) - \tilde{G}(i\sqrt{r})} \right] - \bar{\beta} \log \left[\frac{1 - \overline{\tilde{G}(-i\sqrt{r})} \tilde{G}(\zeta)}{1 - \overline{\tilde{G}(i\sqrt{r})} \tilde{G}(\zeta)} \right]$$

where again we use the principal branch of logarithm.

Note that for the principal branch of logarithm, whenever z and w are both in the right half plane, it holds that $\log zw = \log z + \log w$ with no correction of the branch. So we can write

(4.8)

$$x(\zeta) - x_1(\zeta) = \beta \log \left[\frac{\zeta + i\sqrt{r}}{\zeta - i\sqrt{r}} \cdot \frac{\tilde{G}(\zeta) - \tilde{G}(i\sqrt{r})}{\tilde{G}(\zeta) - \tilde{G}(-i\sqrt{r})} \right] - \bar{\beta} \log \left[\frac{1 - i\sqrt{r}\zeta}{1 + i\sqrt{r}\zeta} \cdot \frac{1 - \overline{\tilde{G}(i\sqrt{r})} \tilde{G}(\zeta)}{1 - \overline{\tilde{G}(-i\sqrt{r})} \tilde{G}(\zeta)} \right].$$

Observe that

$$\lim_{r \rightarrow 0} \left[\frac{\zeta + i\sqrt{r}}{\zeta - i\sqrt{r}} \cdot \frac{\tilde{G}(\zeta) - \tilde{G}(i\sqrt{r})}{\tilde{G}(\zeta) - \tilde{G}(-i\sqrt{r})} \right] = 1$$

so since we are using the principal branch of \log the first term must approach 0 as $r \rightarrow 1$, and similarly for the second term.

Using (4.2) and (4.6) we see that

$$\begin{aligned} m &= 2\pi \operatorname{Re} \left(\beta (x - x_1)|_{-i\sqrt{r}} - \beta (x - x_1)|_{i\sqrt{r}} \right) \\ (4.9) \quad &= 4\pi \operatorname{Re} \left(-\beta^2 \log \frac{\tilde{G}'(i\sqrt{r})i\sqrt{r}}{\tilde{G}(i\sqrt{r})} \right) + \log \left[\frac{1+r}{1-r} \cdot \frac{1 - |\tilde{G}(i\sqrt{r})|^2}{1 + |\tilde{G}(i\sqrt{r})|^2} \right] \end{aligned}$$

where we have repeatedly used $\tilde{G}(-i\sqrt{r}) = -\tilde{G}(i\sqrt{r})$ and $\tilde{G}'(-i\sqrt{r}) = \tilde{G}'(i\sqrt{r})$. Note that the first term goes to 0 as $r \rightarrow 1$ since this holds for (4.8), and the second term uses the principal branch of logarithm. Using

$$\tilde{G}(i\sqrt{r}) = \sqrt{G(-r)} \text{ and } \tilde{G}'(i\sqrt{r}) = \frac{i\sqrt{r}G'(-r)}{\sqrt{G(-r)}}$$

we obtain

$$m = 4\pi \operatorname{Re} \left[-\beta^2 \log \frac{(-r)G'(-r)}{G(-r)} \right] + \log \left[\frac{1+r}{1-r} \frac{1 - |G(-r)|}{1 + |G(-r)|} \right]$$

and by the definition of G , $w_0 = -G(-r)$ and $|f(w_0)| = f(w_0) = r$ we obtain

$$G'(-r) = \frac{1 - |w_0|^2}{(1 - |f(w_0)|^2)f'(0)}$$

and so

$$m = 4\pi \operatorname{Re} \left[\beta^2 \log \left(\frac{w_0 f'(0)}{f(w_0)} \frac{1 - |f(w_0)|^2}{1 - |w_0|^2} \right) \right] + \log \left(\frac{1 + |f(w_0)|}{1 - |f(w_0)|} \frac{1 - |w_0|}{1 + |w_0|} \right)$$

where the branches of logarithm are as claimed. \square

Remark 4.2. By conformal invariance, we can obtain an expression for $m(\mathbb{D}, D_1, Q(z)dz^2)$ for *any* quadratic differential with one simple pole, one double pole, and one double zero on the boundary by composing with disk automorphisms.

Corollary 4.3. *Let $f : \mathbb{D} \rightarrow \mathbb{D}$ be a one-to-one conformal map such that $f(0) = 0$. Let*

$$I(f, w) = 4\pi \operatorname{Re} \left[-e^{i\psi} \log \left(\frac{wf'(0)}{f(w)} \frac{1 - |f(w)|^2}{1 - |w|^2} \right) \right] + \log \left(\frac{1 + |f(w)|}{1 - |f(w)|} \frac{1 - |w|}{1 + |w|} \right)$$

where the branches are determined as in Theorem 4.3. Then $I(f, w) \leq 0$ and equality holds at a point $w \in \mathbb{D}$ if and only if for some θ f maps onto \mathbb{D} minus trajectories of $e^{-2i\theta}Q(e^{i\theta}z)dz^2$ where $Q(z)dz^2$ is given by 4.3 for $r = |f(w_0)|$.

Furthermore, $I(f, w)$ is monotonic in the sense that if $f_i : \mathbb{D} \rightarrow \mathbb{D}$ are one-to-one conformal maps for $i = 1, 2$ such that $f_1(0) = f_2(0) = 0$, $f_1(w) = f_2(w)$ and $f_1(\mathbb{D}) \subseteq f_2(\mathbb{D})$ then $I(f_1, w) \leq I(f_2, w)$.

Proof. Assume first that $f(\mathbb{D})$ is simple. Define θ by $f(w) = re^{i\theta}$. The upper bound of 0 follows from Theorems 4.1 and 3.22, applied to the function $f_\theta(z) = e^{-i\theta}f(z)$, using the fact that $I(f, w) = I(f_\theta, w)$. Equality holds if and only if f_θ maps onto \mathbb{D} minus trajectories of $Q(z)dz^2$; that is, if and only if f maps onto the disk minus trajectories of $e^{-2i\theta}Q(e^{i\theta}z)$. Monotonicity similarly follows from Corollary 3.24.

The general claim follows from Proposition 2.12 and Remark 3.26. \square

Note that we could have also phrased the proof in terms of the quadratic differential $e^{-2i\theta}Q(e^{i\theta}z)dz^2$ rather than applying a rotation to the function.

We now give some special cases of this theorem.

4.3. Pick growth theorems and two-point distortion theorem of Ma and Minda.

Theorem 4.1 implies the classical growth estimates for bounded univalent functions. Choosing $e^{i\psi} = -1$ we obtain

$$(4.10) \quad I_{\text{lower}}(f, w) = \log \left[\frac{|w||f'(0)|}{(1 + |w|)^2} \frac{(1 + |f(w)|)^2}{|f(w)|} \right] \leq 0$$

which after exponentiating becomes

$$|f'(0)| \cdot \frac{|w|}{(1 + |w|)^2} \leq \frac{|f(w)|}{(1 + |f(w)|)^2}$$

which is equivalent to the lower bound in Pick's growth theorem for bounded univalent functions. Similarly, choosing $e^{i\psi} = 1$ we obtain

$$(4.11) \quad I_{\text{upper}}(f, w) = \log \left[\frac{|f(w)|}{(1 - |f(w)|)^2} \cdot \frac{(1 - |w|)^2}{|w||f'(0)|} \right] \leq 0$$

whose exponential gives

$$\frac{|f(w)|}{(1 - |f(w)|)^2} \leq |f'(0)| \cdot \frac{|w|}{(1 - |w|)^2}$$

which is equivalent to the upper bound in Pick's growth theorem. However the result is stronger in that the quantities (4.10) and (4.11) are in fact monotonic in the sense of Theorem 4.1. That is, given one-to-one conformal maps $f_i : \mathbb{D} \rightarrow \mathbb{D}$ such that $f_1(0) = f_2(0) = 0$, $f_1(w) = f_2(w)$, and $f_1(\mathbb{D}) \subseteq f_2(\mathbb{D})$ then $I_{\text{lower}}(f_1, w) \leq I_{\text{lower}}(f_2, w)$, and similarly for I_{upper} .

In fact, we can write the growth theorems in a form due to Ma and Minda (somewhat modified, algebraically). In this case the module m can be written in terms of more familiar

conformal invariants. Let

$$(4.12) \quad \lambda_{\mathbb{D}}(u) = \frac{1}{1 - |z|^2}; \quad d_{\mathbb{D}}(u, v) = \frac{1}{2} \log \frac{1 + |(u - v)/(1 - \bar{v}u)|}{1 - |(u - v)/(1 - \bar{v}u)|}$$

denote the hyperbolic line element and distance functions on \mathbb{D} respectively. Let λ_{D_1} and d_{D_1} denote the hyperbolic line element and distance function on D_1 , which can be explicitly written for a conformal bijection $F : D_1 \rightarrow \mathbb{D}$ as

$$(4.13) \quad \lambda_{D_1}(v) = \frac{|F'(v)|}{1 - |F(v)|^2} \quad \text{and} \quad d_{D_1}(u, v) = d_{\mathbb{D}}(F(u), F(v)).$$

For simply connected Riemann surfaces D and D_1 with hyperbolic metrics, such that $D_1 \subseteq D$, define

$$(4.14) \quad J_{lower}(D, D_1, u, v) = \frac{e^{-4d_{D_1}(u, v)} - 1}{e^{-4d_D(u, v)} - 1} \cdot \frac{\lambda_D(u)}{\lambda_{D_1}(u)} \quad J_{upper}(D, D_1, u, v) = \frac{e^{4d_D(u, v)} - 1}{e^{4d_{D_1}(u, v)} - 1} \cdot \frac{\lambda_{D_1}(u)}{\lambda_D(u)}.$$

Theorem 4.4. *Let D be a conformal disk and $D_1 \subseteq D$ simple. Fix points u and v in D and let $\rho, \tau \in \partial D$ be the terminal points of the geodesic through u and v , arranged in the order ρ, u, v, τ . Let α_{ρ} be the unique quadratic differential admissible for D with a double pole at u , a simple pole at v , and a double zero at ρ and no other zeros or poles in the closure of D . Let α_{τ} be the unique quadratic differential with a double pole at u , a simple pole at v and a double zero at τ and no other zeros or poles. Then for any simple domain D_1 containing u and v ,*

$$m(D, D_1, Q_{\tau}(z)dz^2) = \log J_{lower}(D, D_1, u, v)$$

and

$$m(D, D_1, Q_{\rho}(z)dz^2) = \log J_{upper}(D, D_1, u, v).$$

Proof. Observe that the expressions $J_{lower}(D, D_1, u, v)$ and $J_{upper}(u, v)$ are conformally invariant in the sense that if $g : D \rightarrow E$ is a conformal bijection then $J_{lower}(g(D), g(D_1), g(u), g(v))$. Now observe that if $Q(z)dz^2$ is given by (4.3) with the specific value $e^{i\psi} = -1$, then $\alpha_{\tau} = g^*Q(z)dz^2$ where $g : D \rightarrow \mathbb{D}$ is a conformal bijection such that $g(\tau) = -1$, $g(v) = 0$ and $g(u) = r$. By Theorem 3.19 it thus suffices to prove the claim for $D = \mathbb{D}$, $u = 0$, $v = f(w) = r$ (in which case we will have $g(\tau) = -1$ since g is a hyperbolic isometry). Similarly for I_{upper} .

Let $f : \mathbb{D} \rightarrow D_1$ be a conformal bijection such that $f(0) = 0$. By (4.12) and (4.13) with $F = f^{-1}$ we have that

$$\lambda_{\mathbb{D}}(0) = 1, \quad d_{\mathbb{D}}(0, f(w)) = \frac{1}{2} \log \frac{1 + |f(w)|}{1 - |f(w)|}$$

and

$$\lambda_{D_1}(0) = \frac{1}{|f'(0)|} \quad d_{D_1}(0, f(w)) = \frac{1}{2} \log \frac{1 + |w|}{1 - |w|}.$$

So

$$J_{lower}(\mathbb{D}, D, u, v) = \frac{|w||f'(0)|}{(1 + |w|)^2} \cdot \frac{(1 + |f(w)|)^2}{|f(w)|}$$

and

$$J_{upper}(\mathbb{D}, D, u, v) = \frac{(1 - |w|)^2}{|w||f'(0)|} \cdot \frac{|f(w)|}{(1 - |f(w)|)^2}.$$

This completes the proof. \square

Since simple domains are dense by Proposition 2.12, we have a monotonic, conformally invariant version of the growth theorem of Pick/Ma-Minda.

Corollary 4.5. *Let D and D_1 be simply connected hyperbolic Riemann surfaces with hyperbolic metrics such that $D_1 \subseteq D$ and let $u, v \in D_1$. Then $J_{\text{lower}}(D, D_1, u, v) \leq 0$ and $J_{\text{upper}}(D, D_1, u, v) \leq 0$ with equality if and only if D_1 is D minus trajectories of α_τ or α_ρ respectively. Furthermore, if $D_1 \subseteq D_2 \subseteq \mathbb{D}$ then $J_{\text{upper}}(D_1, u, v) \leq J_{\text{upper}}(D_2, u, v)$ and $J_{\text{lower}}(D_1, u, v) \leq J_{\text{lower}}(D_2, u, v)$.*

Remark 4.6. It is interesting to observe that the general growth Theorem 4.1 interpolates the upper and lower bound in the Pick/Ma-Minda growth theorem, by allowing the zero of the quadratic differential to move between the two ends of the hyperbolic geodesic passing through the points 0 and w .

4.4. Argument estimates. Choosing $\beta^2 = i$ and $\beta^2 = -i$ in Theorem 4.1, we obtain the following monotonic functionals.

Corollary 4.7. *Let $f : \mathbb{D} \rightarrow \mathbb{D}$ be one-to-one and satisfy $f(0) = 0$. Then*

$$-\arg \left(\frac{f(w)}{wf'(0)} \right) + \log \left(\frac{1 + |f(w)|}{1 - |f(w)|} \cdot \frac{1 - |w|}{1 + |w|} \right) \leq 0$$

and

$$\arg \left(\frac{f(w)}{wf'(0)} \right) + \log \left(\frac{1 - |f(w)|}{1 + |f(w)|} \cdot \frac{1 + |w|}{1 - |w|} \right) \leq 0$$

where we use the branch of argument such that $\arg[f(w)/(wf'(0))]$ goes to 0 as $w \rightarrow 0$ and the principal branch of logarithm. Both expressions are monotonic in f in the sense of Corollary 4.3.

The equality statement can be deduced from Corollary 4.3.

5. APPENDIX: CONVERGENCE OF THE CONTOUR INTEGRAL

We need to show that the integral (3.7) converges, by proving Theorem 3.18. We do this, and also show that the definition is natural, with the help of the following parametrization of $\partial\mathbb{D}$. Let $F : \mathbb{D} \rightarrow D_2$ be a conformal map. By Theorem 2.10 $\partial D \setminus \mathfrak{V}$ consists of finitely many analytic arcs B_i , $i = 1, \dots, k$, with endpoints lying in \mathfrak{V} . Each analytic arc is a free boundary arc in the sense of Caratheodory [4, Section 348]. Let $F : \mathbb{D} \rightarrow D$ be a conformal bijection onto D . We will need the following lemma. Although geometrically it is almost obvious, a careful proof involves many details.

Lemma 5.1. *Let D_1 be a conformal disk and D_2 be a simple domain in D_1 . Let \mathfrak{V} be the set of vertices of D_2 , and let $\{B_i\}$ be the connected components of $\partial D_2 \setminus \mathfrak{V}$. Let $F : \mathbb{D} \rightarrow D_2$ be a conformal bijection of \mathbb{D} onto D_2 . F has a continuous extension \hat{F} to $\overline{\mathbb{D}}$. Let $e^{i\theta_1}, \dots, e^{i\theta_m}$ be the elements of $\hat{F}^{-1}(\mathfrak{V})$, arranged so that $\theta_1 < \theta_2 < \dots < \theta_m$. Let $A_j = \{e^{i\theta} : \theta_j < \theta < \theta_{j+1}\}$ (where we set $\theta_{m+1} = \theta_1 + 2\pi$).*

- (1) *For each analytic boundary arc B_i of ∂D_2 , \hat{F} maps precisely one or two of the arcs A_1, \dots, A_m onto B_i ; if it maps two separate arcs onto B_i it does so with opposite orientation.*

- (2) The restriction of \hat{F} to each A_i is a one-to-one analytic parametrization.
- (3) For each A_i , $\pi^{-1} \circ \hat{F}(A_i)$ consists of two disjoint analytic arcs of $\partial\tilde{D}_2$, and each branch of $\pi^{-1} \circ \hat{F}^{-1}$ is an analytic parametrization.
- (4) The collection γ_i of arcs $\pi^{-1} \circ \hat{F}(A_j)$ is a complete set of maximal boundary arcs of $\partial\tilde{D}_2$, with the orientation induced by $\pi^{-1} \circ \hat{F}$.

Proof. By Theorem 2.8, for any $p \in \partial\mathbb{D}$, we can choose a disk B centred on p such that the image of $B \cap \mathbb{D}$ is bounded by a Jordan curve consisting of the analytic arc $F(\partial B \cap \mathbb{D})$ and two analytic trajectories of α . By Carathéodory's theorem [17], F extends continuously to the boundary of $B \cap \mathbb{D}$, and in particular to an open interval arc of $\partial\mathbb{D}$ containing p . This proves that F has a continuous extension \hat{F} to $\partial\mathbb{D}$.

To prove (2), fix A_i and set $U = \{re^{i\theta} : a_i < \theta < a_{i+1} \text{ and } s < r < 1\}$ for some $0 < s < 1$ and $\theta_i < a_i < a_{i+1} < \theta_{i+1}$. Since $\hat{F}(A_i)$ contains no vertices of ∂D_2 , $F(U)$ is bounded by a Jordan curve consisting of four analytic curves. In particular, Carathéodory's theorem implies that the restriction of F to U extends homeomorphically to the boundary of U , and thus is one-to-one on A_i . Since $\hat{F}(A_i)$ is an analytic arc, by the Schwarz reflection principle $F|_U$ extends to a biholomorphism of an open neighbourhood of A_i .

We now prove (1). Since $\hat{F}(A_i)$ joins two vertices of ∂D_2 , it must be a surjection onto some B_j . We show that there is at most two pre-images of any $p \in \partial D \setminus \mathfrak{V}$. There is at least one pre-image of p under \hat{F} . Assume that there are three distinct pre-images, q_1 , q_2 and q_3 say. By the previous paragraph there exists an open disc $B(p; r)$ of p and open neighbourhoods V_i of q_i , $i = 1, \dots, 3$ such that \hat{F} has an extension to a biholomorphism of V_i onto $B(p; r)$. By Theorem 2.8 we may take r small enough that $B(p; r) \setminus \partial D$ consists of precisely two connected components, say W and W_* . By continuity of the extension of \hat{F} we can furthermore choose r small enough that V_i is contained in some disc $B(q_i, r_i)$ such that $r_i < 1$ for $i = 1, \dots, 3$. For each i , the pre-image of either W or W_* is the connected component of $V_i \setminus \partial D$ contained in D ; call this C_i . Thus the original map F takes C_1 , C_2 and C_3 each bijectively onto one of the sets W and W_* . This contradicts the fact that F is one-to-one. Thus there are at most two distinct pre-images of p .

Now assume that \hat{F} maps two arcs A_i and A_j onto B_k say. For any $p \in B_k$, choosing a disk B containing p such that $B \cap D_2$ contains two connected components, since F is orientation preserving we have that $\hat{F}|_{A_i}$ and $\hat{F}|_{A_j}$ endow B_k with opposite orientations. This proves (1).

The claims (3) and (4) follow immediately from the properties of the covering π . □

We may now prove Theorem 3.18.

Proof. We prove both claims simultaneously. Fix γ_i . Let $\pi(\gamma_i) = \hat{F}(A_j)$, say.

Let $p \in \gamma_i$. Since γ_i is analytic there is a biholomorphism G of an open set U containing p onto a disk D in the lower half plane such that $G(\gamma_i \cap U)$ is an interval on the real line; let J be any compact sub-interval containing $G(p)$ in its interior and let I be the compact subarc $G^{-1}(J)$ of γ_i . The set U can be chosen so that π is a biholomorphism of U . We then have that $q_2 \circ G^{-1} = \text{Re}(h)$ for some holomorphic h on D , by the Schwarz reflection principle applied to q_2 . By the Cauchy-Riemann equations and conformal invariance of the integral

(3.5)

$$\int_I q_1 * dq_2 = \int_J q_1 \circ G^{-1} * d(q_2 \circ G^{-1}) = \operatorname{Re} \int_J q_1 \circ G^{-1} \left(\frac{2}{i} \frac{\partial h}{\partial z} dz \right)$$

which exists since h is analytic on J and q_1 . In particular, the integral exists on any compact sub-arc of γ_i .

With notation as in Lemma 5.1, set $\hat{F}(e^{i\theta_p}) = \pi(p)$ and observe that there is a sector

$$S_p = \{re^{i\theta} : \theta_p - \epsilon \leq \theta \leq \theta_p + \epsilon \text{ and } r_p \leq r \leq 1\}$$

such that $\pi^{-1} \circ \hat{F}(S_p)$ is compactly contained in U for one of the choices of π^{-1} . Set $I_r = S_p \cap C_r$. Since $G \circ \pi^{-1} \circ \hat{F}^{-1}$ is a holomorphic function of z on an open neighbourhood of S_p , we have that

$$\begin{aligned} \lim_{r \nearrow 1} \int_{\pi^{-1} \circ \hat{F}^{-1}(I_r)} q_1 * dq_2 &= \lim_{r \nearrow 1} \operatorname{Re} \int_{G \circ \pi^{-1} \circ \hat{F}^{-1}(I_r)} q_1 \circ G^{-1} \frac{2}{i} \frac{\partial h}{\partial z} dz \\ &= \operatorname{Re} \int_{G \circ \pi^{-1} \circ \hat{F}^{-1}(I_1)} q_1 \circ G^{-1} \frac{2}{i} \frac{\partial h}{\partial z} dz \\ &= \int_{\pi^{-1} \circ \hat{F}^{-1}(I_1)} q_1 * dq_2. \end{aligned}$$

On the other hand, if p is an endpoint of a γ_i , it is a vertex. By Theorem 2.8 we can find a map ϕ on the double cover in a neighbourhood U of p so that $\pi^*\alpha$ has the form $w^n dw^2$ in a neighbourhood of p (possibly $n = 0$, if p is a regular point of $\pi^*\alpha$). Thus the trajectories map under ϕ to linear rays in the plane emanating from 0. We may assume that $\phi(U)$ is a disk centred at 0, which is small enough that it contains no other trajectories than the rays. Consider the connected component of $S = \phi(U) \cap \phi(\tilde{D}_2)$ which is bounded by $\phi(\gamma_i)$ such that $\phi(\gamma_i)$ is positively oriented with respect to S (S is a radial segment of a disk). It is bounded by an arc of a circle and another ray which must be $\phi(\gamma_j)$ for some j . At least one γ_j must be such that $\phi(\gamma_j)$ is positively oriented with respect to the segment S , and we choose this one. Finally, choose a biholomorphism H taking S onto a half disk $\Omega = \{z : |z| < r \text{ and } \operatorname{Im}(z) < 0\}$. By Carathéodory's theorem it extends to a homeomorphism of the boundary, and by composing with an automorphism of Ω we can arrange that $\phi(\gamma_i)$ and $\phi(\gamma_j)$ map onto $(0, r)$ and $(-r, 0)$. In summary, the map $G = H \circ \phi$ is a conformal bijection taking a connected component of $U \cap \tilde{D}_2$ onto Ω , and by Schwarz reflection H has an analytic extension to a neighbourhood of $\gamma_i \cap U$ and to a neighbourhood of $\gamma_j \cap U$ (we do not demand that these separate extensions agree). In particular, $q_2 \circ H$ extends continuously to $H(\gamma_i)$ and $H(\gamma_j)$ and equals 0 there. By Schwarz reflection, $q_2 \circ H$ extends to a harmonic function on the full disk $\{z : |z| < r\}$. Choose a subarc I of $\gamma_i \cap U$ with endpoint p , such that $J = H(I)$ is an interval $(0, s)$ or $(-s, 0)$ where $s < r$. Using change of variables

$$\int_I q_1 * dq_2 = \int_J q_1 \circ H * d(q_2 \circ H)$$

which converges since $q_2 \circ H$ has an analytic completion to $|z| < r$ and $q_1 \circ H$ is continuous on the real line. Combining this with the convergence on compact subarcs of the interior, we have shown that the integral converges on γ_i .

Assume now that p is an initial endpoint of γ_i with respect to its orientation. (The other case is similar so we omit it). As above, there is a sector

$$S_p = \{z = re^{i\theta} : r_p \leq r \leq 1 \text{ and } \theta_j \leq \theta \leq \theta_j + \epsilon\}$$

such that $\pi^{-1} \circ \hat{F}(S_p)$ is compactly contained in U for the relevant choice of π^{-1} . Setting $I_r = S_p \cap C_r$ we obtain as above that

$$\lim_{r \nearrow 1} \int_{\pi^{-1} \circ \hat{F}(I_r)} q_1 * dq_2 = \int_{\pi^{-1} \circ \hat{F}(I_1)} q_1 * dq_2.$$

Set C_r^j be the portion of C_r between θ_j and θ_{j+1} . Since γ_i is compact, we have shown that for a single determination of π^{-1} along $\hat{F}(A_i)$

$$\lim_{r \nearrow 1} \int_{\pi^{-1} \circ \hat{F}(C_r^j)} q_1 * dq_2 = \int_{\pi^{-1} \circ \hat{F}(A_j)} q_1 * dq_2$$

Since by part (4) of Lemma 5.1 the set of such γ_i is a complete set of maximal boundary arcs of \tilde{D}_2 , this completes the proof. \square

Remark 5.2. In Nehari's paper [14], the problem of two-sided boundary arcs did not arise, since he assumed that the boundary of the domain was a finite number of closed disjoint analytic arcs. As we have seen, the details in this Appendix allow us to include extremal domains.

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